Randomized $\Omega(n^2)$ Lower Bound for Knapsack

Dima Grigoriev* Marek Karpinski†

Abstract

We prove $\Omega(n^2)$ complexity *lower bound* for the general model of *randomized computation trees* solving the *Knapsack Problem*, and more generally *Restricted Integer Programming*. This is the *first nontrivial* lower bound proven for this model of computation. The method of the proof depends crucially on the new technique for proving lower bounds on the *border complexity* of a polynomial which could be of independent interest.

0 Introduction

We prove for the first time nonlinear lower bounds on the depth of randomized computation trees (RCTs) (see e.g. [MT82], [S83], [M85a], [GKMS96]) recognizing sets like *unions of hyperplanes* (i.e. *linear arrangements*) or *intersections of halfspaces* (polyhedra). As an application we prove a quadratic lower bound on RCTs solving the knapsack problem, or more generally, the restricted integer programming.

Obtaining general lower bounds for randomized computation was an open question for a long time (see e.g. [MT82], [S83], [M85a, b, c], [KV88], [CKKLW95]). Only recently, a nonlinear lower bound was proven in [GKMS96] for a weaker model of randomized d-decision trees (d-RDTs), in which testing polynomials have degrees at most d (for 2-dimensional case the lower bound was proven in [GK93], and for the generic arrangements a lower bound was proved in [GK94]). In particular, for d-RDTs the lower bound $\Omega(n \log n)$ was proven for the *Element Distinctness Problem*, and also the lower bound $\Omega(n^2)$ was proven for the *Knapsack Problem* ([GKMS96]). The main difficulty whith proving lower bounds on RCTs is that the degree of testing polynomials could be possibly exponential. Therefore, we develop in the present paper a new method for obtaining complexity lower bounds for RCTs.

The method developed in the present paper cannot be directly applied for the Element Distinctness Problem. In [BKL93] (cf. also [GKMS96]), a linear depth RCT was constructed for a similar problem (*permutation problem*) $\{(x, y) \in \mathbb{R}^{2n} : y$ is a *permutation* of x$\}$ beating therefore its deterministic $\Omega(n \log n)$ lower bound (cf. [B83]). This example shows that the (still open problem) of complexity of an RCT for the Element Distinctness is quite delicate.

We also mention that a linear $\frac{n}{4}$ lower bound for an RCT recognizing the arrangement $\bigcup_{1 \le i \le n}\{X_i = 0\}$ or the "orthant" $\bigcap_{1 \le i \le n}\{X_i \ge 0\}$ was proved in [GKMS96]. For a stronger model of randomized analytic decision trees ($RADT$) a complexity upper bound $O(\log^2 n)$ for testing $\bigcap_{1 \le i \le n}\{X_i \ge 0\}$ was proven in [GKS96] (for deterministic analytic decision trees the exact complexity bound n was proved in [R72], [MPR94])

For deterministic models of decision and computation trees several methods for obtaining lower bounds were developed earlier. The "topological" methods based on the number of connected components ([SY82], [B83]), or more general, on the sum of Betti numbers ([BLY92], [Y94]), provide the lower bound $\Omega(n^2)$ for the Knapsack Problem and the lower bound $\Omega(n \log n)$ for the Element Distinctness Problem or the Permutation Problem. The already mentioned example from

*Dept. of Computer Science and Mathematics, Penn State University, University Park. Research partially supported by NSF Grant CCR-9424358. Email: dima@cse.psu.edu

†Dept. of Computer Science, University of Bonn, 53117 Bonn, and the International Computer Science Institute, Berkeley. Research partially done while visiting Dept. of Computer Science, Princeton University. Research supported by DFG Grant KA 673/4-1, and by the ESPRIT BR Grants 7097 and EC-US 030 and by DIMACS. Email: marek@cs.uni-bonn.de

[BKL93] shows that these "topological" bounds cannot be directly extended to RCTs.

For testing a polyhedron (for which the topological methods are not applicable), the differential-geometric method (involving the curvature) for obtaining complextity lower bounds for deterministic computations was developped in [GKV95], which provides $\Omega(\log N)$ lower bound for decision trees (see also [GKV95]) and $\Omega(\log N/\log\log N)$ for computation trees, where N is the number of all faces of the polyhedron.

We now briefly describe the content of the paper. In section 1 we introduce the notion of the *border complexity* of a polynomial generalizing the notion of the *border rank* of a tensor, cf. [S90], [B79], [BCLR79], and prove a lower bound on it in terms of the number of connected components, which could be of independent interest.

In section 2 we prove the main theorem which provides a lower bound for an RCT testing an arrangement or a polyhedron. For that purpose we use some tools (in particular, the tree of flags) from [GKMS96], but the proof is different since the degree of RCTs could be exponential as we already mentioned.

In section 3 as an application of the main theorem we give a complexity quadratic lower bound for RCT testing the *Restricted Integer Programming* and in particular, the *Knapsack Problem*.

1 Lower bound on the border complexity

We start now with the technical development leading to the crucial for this paper lower bound on the *border complexity* of a polynomial.

Let $H_1,\dots,H_{n-k}\subset\mathbb{R}^n$ be hyperplanes such that their intersection $\Gamma=H_1\cap\cdots\cap H_{n-k}$ has the dimension $\dim\Gamma=k$. Fix arbitrary coordinates $Z_1,\dots,Z_k$ in Γ. Then treating $H_1,\dots,H_{n-k}$ as the coordinate hyperplanes of the coordinates $Y_1,\dots,Y_{n-k}$, one gets the coordinates $Z_1,\dots,Z_k,Y_1,\dots,Y_{n-k}$ in $\mathbb{R}^n$.

For any polynomial $f\in\mathbb{R}[X_1,\dots,X_n]$ rewrite it in the coordinates $\overline{f}(Z_1,\dots,Z_k,Y_1,\dots,Y_{n-k})$ and following [GKMS96], define its leading term

$$lm(f)=\alpha Z_1^{m_1'}\cdots Z_k^{m_k'}Y_1^{m_1}\cdots Y_{n-k}^{m_{n-k}}$$

$0\neq\alpha\in\mathbb{R}$ (with respect to the coordinate system $Z_1,\dots,Z_k,Y_1,\dots,Y_{n-k}$) as follows. First, take the minimal integer m_{n-k} such that $Y_{n-k}^{m_{n-k}}$ occurs in the terms of f. Consider the polynomial

$$0\not\equiv f^{(1)}=\left(\frac{\overline{f}}{Y_{n-k}^{m_{n-k}}}\right)(Z_1,\dots,Z_k,Y_1,\dots,Y_{n-k-1},0)\in$$
$$\mathbb{R}[Z_1,\dots,Z_k,Y_1,\dots,Y_{n-k-1}]$$

which could be viewed as a polynomial on the hyperplane H_{n-k}. Observe that m_{n-k} depends only on H_{n-k} and not on $Z_1,\dots,Z_k,Y_1,\dots,Y_{n-k-1}$, since a linear transformation of the coordinates $Z_1,\dots,Z_k,Y_1,\dots,Y_{n-k-1}$ changes the coefficients (being the polynomials from $\mathbb{R}[Z_1,\dots,Z_k,Y_1,\dots,Y_{n-k-1}]$) of the expansion of $\overline{f}$ in the variable Y_{n-k}, and a coefficient vanishes identically if and only if it vanishes identically after the transformation. Then $f^{(1)}$ is the coefficient of the expansion of $\overline{f}$ at the power $Y_{n-k}^{m_{n-k}}$.

Second, take the minimal integer m_{n-k-1} such that $Y_{n-k-1}^{m_{n-k-1}}$ occurs in the terms of $f^{(1)}$. In other words, $Y_{n-k-1}^{m_{n-k-1}}$ is the minimal power of Y_{n-k-1} occurring in the terms of $\overline{f}$ in which occurs the power $Y_{n-k}^{m_{n-k}}$. Therefore, m_{n-k}, m_{n-k-1} depend only on the hyperplanes H_{n-k}, H_{n-k-1} and not on $Z_1,\dots,Z_k,Y_1,\dots,Y_{n-k-2}$, since (as above) a linear transformation of the coordinates $Z_1,\dots,Z_k,Y_1,\dots,Y_{n-k-2}$ changes the coefficients (being the polynomials from $\mathbb{R}[Z_1,\dots,Z_k,Y_1,\dots,Y_{n-k-2}]$) of the expansion of $\overline{f}$ in the variables Y_{n-k}, Y_{n-k-1} and a coefficient vanishes identically if and only if it vanishes identically after the transformation. Denote by $0\not\equiv f^{(2)}\in\mathbb{R}[Z_1,\dots,Z_k,Y_1,\dots,Y_{n-k-2}]$ the coefficient of the expansion of $\overline{f}$ at the monomial $Y_{n-k-1}^{m_{n-k-1}}\,Y_{n-k}^{m_{n-k}}$. Obviously

$$f^{(2)}=\left(\frac{f^{(1)}}{Y_{n-k-1}^{m_{n-k-1}}}\right)(Z_1,\dots,Z_k,Y_1,\dots,Y_{n-k-2},0)$$

One could view $f^{(2)}$ as a polynomial on the $(n-2)$-dimensional plane $H_{n-k}\cap H_{n-k-1}$.

Continuing in the similar way, we obtain consecutively the (non-negative) integers $m_{n-k},m_{n-k-1},\dots,m_1$ and the polynomials

$$0\not\equiv f^{(l)}\in\mathbb{R}[Z_1,\dots,Z_k,Y_1,\dots,Y_{n-k-l}]$$

$1 \le l \le n-k$, by induction on l. Herewith, $Y_{n-k-l+1}^{m_{n-k-l+1}}$ is the minimal power of $Y_{n-k-l+1}$ occurring in the terms of $\overline{f}$, in which occurs the monomial $Y_{n-k-l+2}^{m_{n-k-l+2}} \cdots Y_{n-k}^{m_{n-k}}$ for each $1 \le l \le n-k$. Notice that $m_{n-k}, \ldots, m_{n-k-l}$ depend only on the hyperplanes $H_{n-k}, \ldots, H_{n-k-l}$ and not on $Z_1, \ldots, Z_k, Y_1, \ \ldots, Y_{n-k-l-1}$. Then $f^{(l)}$ is the coefficient of the expansion of $\overline{f}$ at the monomial $Y_{n-k-l+1}^{m_{n-k-l+1}} \cdots Y_{n-k}^{m_{n-k}}$ and

$$f^{(l+1)} = \left(\frac{f^{(l)}}{Y_{n-k-l}^{m_{n-k-l}}} \right) (Z_1, \ldots, Z_k, Y_1, \ldots, Y_{n-k-l-1}, 0)$$

Thus, $f^{(l)}$ depends only on $H_{n-k}, \ldots, H_{n-k-l}$ and not on $Z_1, \ldots, Z_k, Y_1, \ldots, Y_{n-k-l-1}$. One could view $f^{(l)}$ as a polynomial on the $(n-l)$-dimensional plane $H_{n-k} \cap \cdots \cap H_{n-k-l+1}$. Continuing, we define also $m_k^{'}, \ldots, m_1^{'}$.

Finally, the leading term $lm(f) = \alpha Z_1^{m_1^{'}} \cdots Z_k^{m_k^{'}} Y_1^{m_1} \cdots Y_{n-k}^{m_{n-k}}$ is the minimal term of $\overline{f}$ in the lexicographical ordering with respect to the ordering $Z_1 > \cdots > Z_k > Y_1 > \cdots > Y_{n-k}$. The leading term $lm(f^{(l)}) = \alpha Z_1^{m_1^{'}} \cdots Z_k^{m_k^{'}} Y_1^{m_1} \cdots Y_{n-k-l}^{m_{n-k-l}}$, we refer to this equality as the maintenance property (see also [GKMS96]).

Denote by $Var(f) = Var^{(H_1, \ldots, H_{n-k})}(f)$ the number of positive (i.e. nonzero) integers among $m_{n-k}, \ldots, m_1$. As we have shown above, $Var(f)$ is independent from the coordinates $Z_1, \ldots, Z_k$ of Γ. Obviously, $Var(f)$ coincides with the number of $1 \le l \le n-k$ such that $Y_{n-k-l} \mid f^{(l)}$, the latter condition is equivalent to that the variety $\{f^{(l)} = 0\} \bigcap (H_{n-k} \cap \cdots \cap H_{n-k-l+1})$ contains the plane $H_{n-k} \cap \cdots \cap H_{n-k-l+1} \cap H_{n-k-l}$ (being a hyperplane in $H_{n-k} \cap \cdots \cap H_{n-k-l+1}$).

It is convenient (see also [GKMS96]) to reformulate the introduced concepts by means of infinitesimals. Namely for a real closed field F (see e.g. [L65]) we say that an element ε transcendental over F is an infinitesimal (relative to F) if $0 < \varepsilon < a$ for any element $0 < a \in F$. This uniquely induces the order on the field $F(\varepsilon)$ of rational functions and further on the real closure $\widetilde{F(\varepsilon)}$ (see [L65]).

One could make the order in $\widetilde{F(\varepsilon)}$ clearer by embedding it in the larger real closed field $F((\varepsilon^{1/\infty}))$ of Puiseux series (cf. e.g. [GV88]). A nonzero Puiseux series has the form $b = \sum_{i \ge i_0} \beta_i \varepsilon^{i/\delta}$, where $-\infty < i_0 < \infty$ is an integer, $\beta_i \in F$ for every integer i; $\beta_{i_0} \ne 0$ and the denominator of the rational exponents $\delta \ge 1$ is an integer. The order on $F((\varepsilon^{1/\infty}))$ is defined as follows: $sgn(b) = sgn(\beta_{i_0})$. When $i_0 \ge 1$, then b is called an infinitesimal, when $i_0 \le -1$, then b is called infinitely large. For any not infinitely large b we define its standard part $st(b) = st_\varepsilon(b) \in F$ as follows: when $i_0 = 0$, then $st(b) = \beta_{i_0}$, when $i_0 \ge 1$, then $st(b) = 0$. In the natural way we extend the standard part to the vectors from $(F((\varepsilon^{1/\infty})))^n$ and further to subsets in this space.

Now let $\varepsilon_1 > \varepsilon_2 > \ldots > \varepsilon_{n+2} > 0$ be infinitesimals, where ε_1 is an infinitesimal relative to $\mathbb{R}$; in general ε_{i+1} is an infinitesimal relative to $\mathbb{R}(\varepsilon_1, \ldots, \varepsilon_i)$ for all $0 \le i \le n+1$. Denote the real closed field $\mathbb{R}_i = \widetilde{\mathbb{R}(\varepsilon_1, \ldots, \varepsilon_i)}$, in particular, $\mathbb{R}_0 = \mathbb{R}$. For an element $b \in \mathbb{R}_{n+2}$ for brevity denote the standard part $st_i(b) = st_{\varepsilon_{i+1}}(st_{\varepsilon_{i+2}} \cdots (st_{\varepsilon_{n+2}}(b) \cdots) \in \mathbb{R}_i$ (provided that it is definable).

Also we will use the Tarski's transfer principle [T51]. Namely, for two real closed fields $F_1 \subset F_2$ a closed (so, without free variables) formula in the language of the first-order theory of F_1 is true over F_1 if and only if this formula is true over F_2.

Tarski's transfer principle implies that a semialgebraic set $\{f_1 \ge 0, \ldots, f_{k_1} \ge 0, f_{k_1+1} > 0, \ldots, f_k > 0\} \subset F^n$, where the polynomials $f_i \in F[X_1, \ldots, X_n]$ have the degrees $deg(f_i) \le d$, has at most $(min\{2^k, (\frac{k}{n})^n\} d^n)^{O(1)}$ connected components (cf. [GV88]), relying on this bound in case $F = \mathbb{R}$ from [W68] (cf. also [BPR94]), which strenghtens the result of [M64].

Another application of Tarski's transfer principle is the concept of the completion. Let $F_1 \subset F_2$ be real closed fields and Ψ be a formula (with quantifiers and, perhaps, with n free variables) of the language of the first-order theory of the field F_1. Then Ψ determines a semialgebraic set $V \subset F_1^n$. The completion $V^{(F_2)} \subset F_2^n$ is a semialgebraic set determined by the same formula Ψ (obviously, $V \subset V^{(F_2)}$). Tarski's transfer principle entails, in particular, that the number of connected components of V is the same as the one of $V^{(F_2)}$ (cf. [GV88]).

One could easily see that for any point $(z_1, \ldots, z_k) \in \mathbb{R}_{k+2}^k$ such that $f^{(n-k)}(z_1, \ldots, z_k) \ne 0$ (we utilize the

introduced above notations) the following equality for the signs

$$\sigma_1^{m_1} \ldots \sigma_{n-k}^{m_{n-k}} sgn\left(f^{(n-k)}(z_1,\ldots,z_k)\right) =$$

$$sgn\left(\overline{f}(z_1,\ldots,z_k,\sigma_1\varepsilon_{k+3},\ldots,\sigma_{n-k}\varepsilon_{n+2})\right) \qquad (1)$$

holds for any $\sigma_1,\ldots,\sigma_{n-k} \in \{-1,1\}$. For any $1 \le i \le n-k$ such that $m_i = 0$ (1) holds also for $\sigma_i = 0$, agreeing that $0^0 = 1$. Moreover, the following polynomial identity holds:

$$f^{(n-k)}(Z_1,\ldots,Z_k) = st_{k+2}\left(\frac{\overline{f}(Z_1,\ldots,Z_k,\varepsilon_{k+3},\ldots,\varepsilon_{n+2})}{\varepsilon_{k+3}^{m_1}\cdots\varepsilon_{n+2}^{m_{n-k}}}\right) \qquad (2)$$

For a family of hyperplanes $H_1,\ldots,H_m \subset \mathbb{R}^n$ let $S = \cup_{1\le i\le m} H_i$ be an arrangement, by $B_0(H_1,\ldots,H_m)$ we denote the number of connected components of the complement $\mathbb{R}^n - S$.

Following e.g. [S90] we define the complexity $s = C(f)$ of a polynomial $f \in \mathbb{R}[X_1,\ldots,X_n]$ as the length of the shortest straight-line program which computes f. Recall that the latter is a sequence of operations $u_1 = X_1,\ldots,u_n = X_n$, then for every $n < j \le s+n$ $u_j = \tilde{u}_{j_1} \odot \tilde{u}_{j_2}$, where for each $i = 1,2$ either $\tilde{u}_{j_i} = u_{j_i}$ with $j_i < j$ or $\tilde{u}_{j_i} \in \mathbb{R}$ and either $\odot = \times$ or $\odot = +$. To every u_j by recursion on j one attaches in the natural way a polynomial $U_j \in \mathbb{R}[X_1,\ldots,X_n]$ (the value of u_j). The straight-line program computes f if $U_{s+n} = f$.

Observe that one could consider also the division $\odot = /$ and the resulting rational functions, but since we deal only with the signs of the testing functions in the computation trees (see below), we could consider separately the computations of the numerators and denominators of the rational functions by means of the straight-line programs without the divisions.

For a polynomial $g \in \mathbb{R}[Z_1,\ldots,Z_k]$ its *border complexity* $\overline{C}(g)$ (cf. [S90] for the notion of the *border rank*) is the *minimal* $C(f)$ where $f \in \mathbb{R}[X_1,\ldots,X_n]$ for a certain $n \ge k$ such that $g = f^{(n-k)}$, for suitable coordinates $Z_1,\ldots,Z_k,Y_1,\ldots,Y_{n-k}$, which we treat as the linear forms in $X_1,\ldots,X_n$.

The main result of this section is the following lower bound on the border complexity.

Proposition: Let for a polynomial $g \in \mathbb{R}[Z_1,\ldots,Z_k]$ its border complexity $\overline{C}(g) \le s$. Assume that $H_1,\ldots,H_m \subset \mathbb{R}^k$ are pairwise distinct hyperplanes such that the corresponding linear functions $L_{H_i} \mid g$, $1 \le i \le m$ (where the zero set of L_{H_i} is H_i). Then $B_0(H_1,\ldots,H_m) \le 2^{O(s+k)}$.

Proof: Let $u_i = X_i$, $1 \le i \le n$; $u_j = \tilde{u}_{j_1} \odot \tilde{u}_{j_2}$, $n+1 \le j \le n+s$ be a straight-line program which computes a certain polynomial $f \in \mathbb{R}[X_1,\ldots,X_n]$ such that $g = f^{(n-k)}$ for suitable coordinates $Z_1,\ldots,Z_k,Y_1,\ldots,Y_{n-k}$ (we utilize the introduced above notations). Express $X_i = \alpha_1^{(i)}Z_1 + \cdots + \alpha_k^{(i)}Z_k + \beta_1^{(i)}Y_1 + \cdots \beta_{n-k}^{(i)}Y_{n-k}$, $1 \le i \le n$, where $\alpha_j^{(i)}, \beta_j^{(i)} \in \mathbb{R}$.

Due to (2) for any point $(z_1,\ldots,z_k) \in \mathbb{R}_2^k$ we have

$$g(z_1,\ldots,z_k) = st_2\left(\frac{\overline{f}(z_1,\ldots,z_k,\varepsilon_{k+3},\ldots,\varepsilon_{n+2})}{\varepsilon_{k+3}^{m_1}\cdots\varepsilon_{n+2}^{m_{n-k}}}\right) \qquad (3)$$

Denote $u'_i = \alpha_1^{(i)}z_1 + \cdots + \alpha_k^{(i)}z_k + \beta_1^{(i)}\varepsilon_{k+3} + \cdots + \beta_{n-k}^{(i)}\varepsilon_{n+2}$, $1 \le i \le n$. Introduce a new variable Z_0 and two semialgebraic sets

$$\mathcal{V} = \Big\{ (z_0,z_1,\ldots,z_k,u_{n+1},\ldots,u_{n+s}) \in \mathbb{R}_{n+2}^{k+s+1} : u_j = \tilde{u}'_{j_1} \odot \tilde{u}'_{j_2}, n+1 \le j \le n+s, \text{ where for each } i = 1,2 \text{ either } \tilde{u}'_{j_i} = u'_{j_i} \text{ when } 1 \le j_i \le n \text{ and } \tilde{u}'_{j_i} = u_{j_i} \text{ when } n < j_i < j, \text{ or } \tilde{u}'_{j_i} \in \mathbb{R} \text{ according to the straight-line program which computes } f; \left(\left(\frac{u_{n+s}}{\varepsilon_{k+3}^{m_1}\cdots\varepsilon_{n+2}^{m_{n-k}}}\right)^2 - \varepsilon_1\right)^2 + \left(z_0^2 + z_1^2 + \cdots + z_k^2 - \frac{1}{\varepsilon_1}\right)^2 < \varepsilon_2 \Big\};$$

$$V = \left\{ (z_0,z_1,\ldots,z_k) \in \mathbb{R}_1^{k+1} : g^2(z_1,\ldots,z_k) = \varepsilon_1; z_0^2 + z_1^2 + \cdots + z_k^2 = \frac{1}{\varepsilon_1} \right\}$$

Denote by $\prod : \mathbb{R}_{n+2}^{k+s+1} \to \mathbb{R}_{n+2}^{k+1}$ the linear projection along the coordinates $u_{n+1},\ldots,u_{n+s}$. The linear projection $\prod : \mathcal{V} \widetilde{\to} \prod(\mathcal{V})$ is an isomorphism of the semialgebraic sets, since the projection

$$\prod(\mathcal{V}) = \Big\{ (z_0,z_1,\ldots,z_k) \in \mathbb{R}_{n+2}^{k+1} :$$

$$\left(\left(\frac{\overline{f}(z_1,\ldots,z_k,\varepsilon_{k+3},\ldots,\varepsilon_{n+2})}{\varepsilon_{k+3}^{m_1}\cdots\varepsilon_{n+2}^{m_{n-k}}}\right)^2-\varepsilon_1\right)^2+$$

$$(z_0^2+z_1^2+\cdots+z_k^2-\frac{1}{\varepsilon_1})^2<\varepsilon_2\Bigg\}$$

and the inverse mapping is given by the polynomial mapping $u_j=\tilde{u}'_{j_1}\odot\tilde{u}'_{j_2}$, $n+1\le j\le n+s$.

Then $V\subset\prod(\mathcal{V})$ because of (3).

Furthermore, $st_1(\prod(\mathcal{V}))=V$; the left side is definable since for any point $(z_0,\ldots,z_k)\in\prod(\mathcal{V})$ the square of its euclidean norm $\|z_o,\ldots,z_k\|^2=z_0^2+\cdots+z_k^2<\frac{1}{\varepsilon_1}+\varepsilon_2^{\frac{1}{2}}<\frac{1}{\varepsilon_1}+1$. By the same reason lemma 1 from [GV88] states that the number N_3 of the connected components of V does not exceed the number N_4 of the connected components of $\prod(\mathcal{V})$, the latter coincides with the number of the connected components of $\mathcal{V}$ since it is isomorphic to $\prod(\mathcal{V})$.

We claim that for any connected component $W\subset\mathbb{R}^k$ (which is an open set in the euclidean topology) of the component $\mathbb{R}^k-\{g=0\}$ and an arbitrary point $w_0\in\partial W$ on the boundary, there exists a point $(z_1,\ldots,z_k)\in W^{(\mathbb{R}_1)}\subset\mathbb{R}_1^k$ from the completion $W^{(\mathbb{R}_1)}$ (as we have seen above from Tarski's transfer principle, the connected components W of the complement are in the bijective correspondence with their completions $W^{(\mathbb{R}_1)}\supset W$, being the connected components of the complement $\{g=0\}^{(\mathbb{R}_1)}$ in $\mathbb{R}_1^k$, the number of these connected components we denote by N_0) such that $g^2(z_1,\ldots,z_k)=\varepsilon_1$ and $st_0(z_1,\ldots,z_k)=w_0$ (cf. lemma 3 from [GV88]). Indeed, pick out an arbitrary point $w\in W$. Taking into account that $w_0\in\partial(W^{(\mathbb{R}_1)})$, so $g(w_0)=0$, and $0<g^2(w)\in\mathbb{R}$ we conclude that g^2 attains on $W^{(\mathbb{R}_1)}$ any intermediate value from $\mathbb{R}_1$ between 0 and $g^2(w)$ (using Tarski's transfer principle), in particular, ε_1. Now take a point $w_1\in W^{(\mathbb{R}_1)}$ being the nearest to w_0 such that $g^2(w_1)=\varepsilon_1$ (its existence follows again from Tarski's transfer principle). It suffices to prove that $st_0(w_1)=w_0$. Suppose the contrary. Then there exists $0<r\in\mathbb{R}$ such that for any point $w_2\in W^{(\mathbb{R}_1)}$ with the distance $\|w_0-w_2\|\le r$ the inequality $g^2(w_2)<\varepsilon_1$ holds. Since $w_0\in\partial W$ there exists a point $w_3\in W$ such that $\|w_0-w_3\|\le r$, then $0<g^2(w_3)\in\mathbb{R}$ and we get a contradiction with the supposition, and that proves the claim.

Furthermore, since $w_0\in\mathbb{R}^k$ and $st_0(z_1,\ldots,z_k)=w_0$, there exists $0<r_1\in\mathbb{R}$ such that the norm $\|z_1,\ldots,z_k\|\le r_1$, a fortiori $\|z_1,\ldots,z_k\|^2\le\frac{1}{\varepsilon_1}$.

Consider a semialgebraic set

$$V_0=\Big\{(z_1,\ldots,z_k)\in\mathbb{R}_1^k:g^2\,(z_1,\ldots,z_k)=\varepsilon_1\Big\}$$

Denote by N_1 the number of the connected components of V_0 containing a point w_4 with the square of the euclidean norm $\|w_4\|^2\le\frac{1}{\varepsilon_1}$. The proved above claim states that the number N_0 does not exceed N_1, taking into account that

$$V_0\subset\Big(\mathbb{R}^k-\{g=0\}\Big)^{(\mathbb{R}_1)}=\mathbb{R}_1^k-(\{g=0\})^{(\mathbb{R}_1)}$$

On the other hand, $B_0(H_1,\ldots,H_m)\le N_0$, since $\sqcap_{1\le i\le m}L_{H_i}\mid g$ (evidently, in every connected component, being an open set in the euclidean topology, of the complement of the arrangement $\Big(\mathbb{R}^k-\bigcup_{1\le i\le m}H_i\Big)\supset\Big(\mathbb{R}^k-\{g=0\}\Big)$, there exists a point at which g does not vanish).

Obviously, N_1 is less than or equal to the number N_2 of the connected components of the set

$$V_1=V_0\cap\Big\{(z_1,\ldots,z_k)\in\mathbb{R}_1^k:\|z_1,\ldots,z_k\|^2\le\frac{1}{\varepsilon_1}\Big\}$$

In its turn $V_1=\prod_0(V)$, where $\prod_0:\mathbb{R}_1^{k+1}\to\mathbb{R}_1^k$ is the projection along the coordinate Z_0. Hence $N_2\le N_3$.

Gathering the obtained chain of inequalities $B_0(H_1,\ldots,H_m)\le N_0\le N_1\le N_2\le N_3\le N_4$ for the numbers of the connected components, we conclude that $B_0(H_1,\ldots,H_m)$ does not exceed the number of connected components of $\mathcal{V}$. The latter is less than $2^{O(s+k)}$ according to [W68] and Tarski's transfer principle (see above).

The proposition is proved.

2 Lower bounds for randomized computation trees

Recall (see e.g. [B83]) that in the computation tree (CT) testing polynomials are computed along paths using the elementary arithmetic operations. In particular, for a testing polynomial $f_i\in\mathbb{R}[X_1,\ldots,X_n]$ at the level

i (assuming that the root has the zero level) we have $C(f_i) \leq i$. Under RCT (cf. [MT82], [S83], [M85a,b,c]) we mean a collection of CT $T = \{T_\alpha\}$ and a probabilistic vector $p_\alpha \geq 0$, $\sum_\alpha p_\alpha = 1$ such that T_α is chosen with the probability p_α. The main requirement is that for any input RCT gives a correct output with the probability $1 - \gamma > \frac{1}{2}$ (γ is called the error probability of RCT).

For a hyperplane $H \subset \mathbb{R}^n$ by $H^+ \subset \mathbb{R}^n$ denote the closed halfspace $\{L_H \geq 0\}$, where L_H is a certain linear function with the zero set H. For a family of hyperplanes $H_1, \ldots, H_m$ the intersections $S^+ = \cap_{1 \leq i \leq m} H_i^+$ is called a polyhedron. An intersection $\Gamma = H_{i_1} \cap \cdots \cap H_{i_{n-k}}$ is called k-face of S^+ if $dim\ \Gamma = dim(\Gamma \cap S^+) = k$. By $\phi_k(S^+)$ we denote the number of k-faces of S^+. Similary (and even simpler) for the arrangement $S = \cup_{1 \leq i \leq m} H_i$ its k-face is any k-dimensional intersection of the form $\Gamma = H_{i_1} \cap \cdots \cap H_{i_{n-k}}$. By $\phi_k(S)$ we denote the number of k-faces of S.

Now we are able to formulate the main result of this paper.

Theorem: Let there exist positive constants c_1, c_2, c_3, c_4 such that $c_3(1 - c_1) < c_2$ and an arrangement $\mathcal{S} = S = \cup_{1 \leq i \leq m} H_i$ or a polyhedron $\mathcal{S} = S^+ = \cap_{1 \leq i \leq m} H_i^+$ satisfy the following properties:

1. $\phi_{[c_1 n]}(\mathcal{S}) \geq \Omega(m^{c_2 n})$;
2. for any k-face Γ of $\mathcal{S}$ with $k \geq c_1 n$ and any subfamily $H_{i_1}, \ldots, H_{i_q}$ of $H_1, \ldots, H_m$ with at least $q \geq m^{c_3}$ hyperplanes such that $H_{i_j} \not\supset \Gamma$ for each $1 \leq j \leq q$ and the hyperplanes $H_{i_1} \cap \Gamma, \ldots, H_{i_q} \cap \Gamma$ in Γ are pairwise distinct, the number of the connected components $B_0^{(\Gamma)}(H_{i_1} \cap \Gamma, \ldots, H_{i_q} \cap \Gamma)$ of the complement in Γ of the arrangement $\cup_{1 \leq j \leq q}(H_{i_j} \cap \Gamma)$ is greater than $\Omega(m^{c_4 n})$.

Then for any RCT recognizing $\mathcal{S}$, its depth is greater than $\Omega(n \log m)$.

Before proceeding to the proof of the theorem, we need some preparation.

First we fix the canonical representation of k-face Γ in two cases: namely, of S and of S^+, respectively (see [GKMS96]). In the case of S take the maximal $i_{n-k} \leq m$ such that $H_{i_{n-k}} \supset \Gamma$, then the maximal i_{n-k-1} such that $H_{i_{n-k-1}} \supset \Gamma$ and $dim(H_{i_{n-k}} \cap H_{i_{n-k-1}}) = n - 2$ (obviously $i_{n-k-1} < i_{n-k}$) and so on we produce the indices $i_{n-k} > i_{n-k-1} > \cdots > i_1$ such that $\Gamma = H_{i_{n-k}} \cap \cdots \cap H_{i_1}$. As the representation of Γ we take the flag of planes: $H_{i_{n-k}} \supset H_{i_{n-k}} \cap H_{i_{n-k-1}} \supset \cdots \supset H_{i_{n-k}} \cap \cdots \cap H_{i_1} = \Gamma$.

Now consider the case of S^+. One can prove (see [GKMS96]) that for any k-face Γ there exists a flag which we treat as a canonical representation of Γ:

$$H_{i_{n-k}} \supset H_{i_{n-k}} \cap H_{i_{n-k-1}} \supset \cdots \supset H_{i_{n-k}} \cap \cdots \cap H_{i_1} = \Gamma$$

such that for each $1 \leq l \leq k$ $H_{i_{n-k}} \cap \cdots \cap H_{i_{n-k-l+1}}$ is $(n-l)$-face of S^+ (the recursion on l implies that $dim(H_{i_{n-k}} \cap \cdots \cap H_{i_{n-k-l+1}}) = n - l$). Moreover, this sequence of indices $i_{n-k} > \cdots > i_1$ is the maximal with respect to the lexicographical ordering (similar to the case of S above) satisfying the latter property.

Fix k-face Γ of $\mathcal{S}$, where either $\mathcal{S} = S$ or $\mathcal{S} = S^+$. Let $H_{i_{n-k}} \supset H_{i_{n-k}} \cap H_{i_{n-k-1}} \supset \cdots \supset H_{i_{n-k}} \cap \cdots \cap H_{i_1} = \Gamma$ be a flag which represents Γ as described above. For a family of polynomials $f_1, \ldots, f_s \in \mathbb{R}[X_1, \ldots, X_n]$ we define $Var^{(\Gamma)}(f_1, \ldots, f_s)$ to be the number of the variables among $Y_1, \ldots, Y_{n-k}$ (we utilize the notations introduced in section 1) which occur in at least one of $lm(f_1), \ldots, lm(f_s)$, where $H_{i_1}, \ldots, H_{i_{n-k}}$ are the coordinate hyperplanes of the coordinates $Y_1, \ldots, Y_{n-k}$, respectively. Since $lm(f_1 \cdots f_s) = lm(f_1) \cdots lm(f_s)$ we get that $Var^{(H_{i_1}, \ldots, H_{i_{n-k}})}(f_1 \cdots f_s) = Var^{(\Gamma)}(f_1 \cdots f_s) = Var^{(\Gamma)}(f_1, \ldots, f_s)$.

For any CT T_1 we denote by $Var^{(\Gamma)}(T_1) = Var^{(H_{i_1}, \ldots, H_{i_{n-k}})}(T_1)$ the maximum of $Var^{(\Gamma)}(f_1 \cdots f_s)$ taken over all the paths of T_1, where $f_1, \ldots, f_s$ are testing polynomials along the path.

The following lemma was proved in [GKMS96] (see also [GKMS96]).

Lemma 1: Let $T = \{T_\alpha\}$ be an RCT recognizing

a) an arrangement $S = \cup_{1 \leq i \leq m} H_i$ such that $\Gamma = \cap_{1 \leq j \leq n-k} H_{i_j}$ is k-face of S, or

b) a polyhedron $S^+ = \cap_{1 \leq i \leq m} H_i^+$ such that for each

$1 \le l \le n-k$ $\cap_{l \le j \le n-k} H_{i_j}$ is $(k+l-1)$-face of S^+ (denote $\Gamma = \cap_{1 \le j \le n-k} H_{i_j}$)

with error probability $\gamma < \frac{1}{2}$. Then $Var^{(H_{i_1},\ldots,H_{i_{n-k}})}(T_\alpha) \ge (1-2\gamma)^2(n-k)$ for a fraction of $\frac{1-2\gamma}{2-2\gamma}$ of all T_α's.

Remark: Notice that the conditions in a), b) are fulfilled if $H_{i_{n-k}} \supset H_{i_{n-k}} \cap H_{i_{n-k-1}} \supset \cdots \supset H_{i_{n-k}} \cap \cdots \cap H_{i_1} = \Gamma$ is the canonical flag representation of Γ in both cases of S and S^+ (see above).

An analogue of lemma 2 from [GKMS96] (see also [GKMS96]) is the following lemma.

Lemma 2: Let $\mathcal{S} = S$ or $\mathcal{S} = S^+$ satisfy the condition 2. of the theorem. Assume that CT T' for some constant $c > 0$, satisfies the inequality $Var^{(\Gamma)}(T') \ge c(1-c_1)n$ for at least M $\lceil c_1 n \rceil$-faces Γ of $\mathcal{S}$. Then the depth t of T' fulfils either $t \ge \Omega(n \log m)$ or $M \le O(3^t m^{(1-c+c_3+\delta)(1-c_1)n})$, where a constant $\delta > 0$ could be made as close to zero as desired.

The proof of lemma 2 differs from the proof of the analogous lemma 2 from [GKMS96] proved for *d-decision* trees, in which the degrees of the testing polynomials do not exceed d, rather than *computation* trees (considered in the present paper), in which the degrees of the testing polynomials could be exponential in the depth t of CT. Therefore the main tool in the proof of lemma 2 is the lower bound on the border complexity from the proposition (see section 1).

Before proving lemma 2 we show how to deduce the theorem from lemmas 1 and 2. Consider RCT $\{T_\alpha\}$ recognizing $\mathcal{S}$ with error probability $\gamma < \frac{1}{2}$. Denote $k = \lceil c_1 n \rceil$. Lemma 1, condition 1. of the theorem and counting imply the existence of T_{α_0} such that the inequality $Var^{(\Gamma)}(T_{\alpha_0}) \ge (1-2\gamma)^2(n-k)$ is true for $M = \frac{1-2\gamma}{2(1-\gamma)}\Omega(m^{c_2 n})$ k-faces Γ of $\mathcal{S}$. Apply lemma 2 to CT $T' = T_{\alpha_0}$ with $c = (1-2\gamma)^2$. If $t \ge \Omega(n \log m)$ the theorem is proved, else since the error probability γ could be made a positive constant as close to zero as desired at the expence of increasing by a constant factor the depth of RCT [M85a,c], take γ such that $(1-c+\delta) < \frac{c_2 - c_3(1-c_1)}{1-c_1}$. Then lemma 2 entails that $t \ge \Omega(n \log m)$, which proves the theorem. Thus, it remains to prove lemma 2.

Proof of lemma 2: To each k-face Γ of $\mathcal{S}$ satisfying the inequality $Var^{(\Gamma)}(T') \ge c(n-k)$, we correspond a path in T' with the testing polynomials $f_1, \ldots, f_s \in \mathbb{R}[X_1, \ldots, X_n]$ such that $Var^{(\Gamma)}(f_1 \cdots f_s) \ge Var^{(\Gamma)}(T')$. Denote $f = f_1 \cdots f_s$. Consider a canonical representation of Γ by a flag (see above)

$$H_{i_{n-k}} \supset H_{i_{n-k}} \cap H_{i_{n-k-1}} \supset \ldots \supset H_{i_{n-k}} \cap \ldots \cap H_{i_1} = \Gamma$$

Fix this path of T' for the time being and consider all k-faces Γ to which corresponds this path. We arrange the representing flags of all these k-faces in a tree $\mathcal{T}$ which we call the tree of flags (cf. the proof of lemma 2 from [GKMS96]). $\mathcal{T}$ has a root with the zero level, each its path has the same length $n-k$ (such trees are called regular), some of its vertices are labeled.

We construct $\mathcal{T}$ by induction on the level of its vertices. The base of induction. For each k-face Γ to which corresponds the fixed path of T', construct a vertex, being a son of the root of $\mathcal{T}$, and to this vertex (of level 1) attach the hyperplane $H_{i_{n-k}}$ (we utilize inroduced above notations). Thus, to different sons of the root different hyperplanes are attached. We label the constructed vertex if and only if $Y_{n-k}|f$ (the latter means that the linear function or the variable Y_{n-k} gives a contribution into $Var^{(\Gamma)}(f)$). Besides, we assign to the constructed vertex the polynomial $f^{(1)} \in \mathbb{R}[Z_1, \ldots, Z_k, Y_1, \ldots, Y_{n-k-1}]$ (see section 1).

Now assume by induction on l that $l < n-k$ levels of $\mathcal{T}$ are already constructed. Consider any vertex v of $\mathcal{T}$ at l-th level. To the vertex v leads the partially labeled path (from the root), to whose vertices the beginning elements of a flag are attached successively:

$$H_{i_{n-k}} \supset H_{i_{n-k}} \cap H_{i_{n-k-1}} \supset \ldots$$
$$\cdots \supset H_{i_{n-k}} \cap \ldots \cap H_{i_{n-k-l+1}} = \Gamma_1$$

Finally, the polynomial $f^{(l)} \in \mathbb{R}[Z_1, \ldots, Z_k, Y_1, \ldots, Y_{n-k-l}]$ is assigned to the vertex v. Recall (see section 1) that $f^{(l)}$ is defined on $(n-l)$-plane Γ_1. Besides, v is either labeled or not labeled.

Thus, to different vertices at the level l are attached the different beginnings of flags.

At the inductive step we construct the sons of v. Namely, for any k-face Γ with the same beginning (4) of its representing flag consider the next element of its flag, let it be $\Gamma_1 \cap H_{i_{n-k-l}}$. Construct a son of v to which we attach $\Gamma_1 \cap H_{i_{n-k-l}}$ and assign the polynomial $f^{(l+1)} \in \mathbb{R}[Z_1, \ldots, Z_k, Y_1, \ldots, Y_{n-k-l-1}]$. We label this vertex if and only if $Y_{n-k-l} | f^{(l)}$ (recall that due to the maintainance property, see section 1, the latter condition means that the linear form or the variable Y_{n-k-l} gives a contribution into $Var^{(\Gamma)}(f)$).

This completes the inductive construction of $\mathcal{T}$. The leaves (or paths) of $\mathcal{T}$ correspond bijectively to k-faces of $\mathcal{S}$ to which the fixed path of T' corresponds. To each leaf (or path) of $\mathcal{T}$ which corresponds to k-face Γ the flag representing Γ $H_{i_{n-k}} \supset H_{i_{n-k}} \cap H_{i_{n-k-1}} \supset \ldots \supset H_{i_{n-k}} \cap \ldots \cap H_{i_1} = \Gamma$ is attached along the path (which is partially labeled).

Now we proceed to estimating the number of leaves of $\mathcal{T}$. For a vertex v consider all its labeled sons (we utilize the introduced above notations). Each labeled son corresponds to a hyperplane H_i such that the linear function $L_{\Gamma_1 \cap H_i} | f^{(l)}$, where $L_{\Gamma_1 \cap H_i}$ is a certain linear function on $(n-l)$-plane Γ_1 with the zero set $\Gamma_1 \cap H_i$, being a hyperplane in Γ_1, and to different sons correspond different hyperplanes $\Gamma_1 \cap H_i$. Consider the family $\mathcal{H}$ of all such hyperplanes H_i. First assume that it contains at least m^{c_3} hyperplanes. Then the condition 2. of the theorem implies that the number of the connected components $b = B_0^{(\Gamma_1)}(\{H_i \cap \Gamma_1\}_{H_i \in \mathcal{H}})$ of the complement in Γ_1 of the arrangement $\bigcup_{H_i \in \mathcal{H}} (H_i \cap \Gamma_1)$ is greater than $\Omega(m^{c_4 n})$. On the other hand the proposition (see section 1) entails that $b \leq 2^{O(s+n-l)} \leq 2^{O(s+n)}$, taking into account that the complexity $C(f) = C(f_1 \cdots f_s) \leq 2s - 1$. This provides the lower bound on the depth of T', namely, $t \geq s \geq \Omega(n \log m)$, that proves lemma 2. Thus, we can assume that any vertex v of $\mathcal{T}$ has less than m^{c_3} labeled sons. Besides the labeled sons, each vertex could have at most m unlabeled sons. Furthermore, due to the maintenance property, along each path of $\mathcal{T}$ at least $c(1-c_1)n$ vertices are labeled (see the inductive step above).

To estimate the number of leaves in $\mathcal{T}$ introduce an auxiliary magnitude $M(R, Q)$ to be the maximal possible number of the leaves in a regular tree (actually, we could stick with subtrees of $\mathcal{T}$, so they are partially labeled) with the length of any path equal to R and with at most Q unlabeled vertices along the path. One can assume w.l.o.g. that $Q \leq R \leq m$ (if $Q > R$ then set $M(R, Q) = 0$, the inequality $R \leq m$ holds since we consider the subtrees of $\mathcal{T}$, and to each path of $\mathcal{T}$ a flag of the length at most m is attached). Considering such a tree and its subtrees with the roots being the sons (both unlabeled and labeled) of the root of the tree, we get the following inductive inequality:

$$M(R, Q) \leq m \cdot M(R-1, Q-1) + m^{c_3} M(R-1, Q)$$

For the base of induction, obviously $M(0, 0) = 1$. By induction on R we obtain the bound $M(R, Q) \leq \beta \cdot m^Q \cdot m^{(c_3+\delta_1)R}$ for arbitrary $\delta_1 > 0$ and a suitable $\beta > 0$.

Substituting $R = n - \lceil c_1 n \rceil$, $Q = (1-c)(n - \lceil c_1 n \rceil)$, we conclude that the number of the leaves of $\mathcal{T}$ is less than $O(m^{(1-c)(1-c_1)n + (c_3+\delta)(1-c_1)n})$ for arbitrary $\delta > 0$.

To complete the proof of lemma 2 it remains to notice that the tree of flags $\mathcal{T}$ was constructed for a fixed path of CT T'; there are at most 3^t paths of T'. On the other hand, every k-face Γ of $\mathcal{S}$, satisfying the inequality $Var^{(\Gamma)}(T') \geq c(1-c_1)n$, corresponds to one of the leaves of a tree of flags constructed for one of the paths of T'. Hence the number of such k-faces $M \leq O(3^t \cdot m^{(1-c+c_3+\delta)(1-c_1)n})$.

3 Quadratic complexity lower bound for RCTs solving the restricted integer programming

The restricted integer programming is the arrangement

$$L_{n,j} = \bigcup_{a \in \{0, \ldots, j-1\}^n} \{aX = 1\} \subset \mathbb{R}^n$$

of $m = j^n$ hyperplanes for some $j \geq 2$ (see e.g. [M85b]). For $j = 2$ $L_{n,2}$ is the knapsack problem.

As an application of the theorem we prove the following corollary.

Corollary: For any RCT solving the restricted integer programming $L_{n,j}$, its depth is greater than $\Omega(n^2 \log j)$.

4 Open Problem

We were not able to prove any superlinear lower bound or a linear upper bound on the *Element Distinctness*(cf. [M85a], [GKMS96]) for randomized computational trees. Note that the corresponding lower bound for randomized decision trees is $\Omega(n \log n)$, [GKMS96].

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