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EMBEDDING POINT SETS INTO PLANE GRAPHS OF SMALL DILATION

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Let S be a set of points in the plane. What is the minimum possible dilation of all plane graphs that contain S? Even for a set S as simple as five points evenly placed on the circle, this question seems hard to answer; it is not even clear if there exists a lower bound > 1. In this paper we provide the first upper and lower bounds for the embedding problem.
1. Each finite point set can be embedded into the vertex set of a finite triangulation of dilation ≤ 1.1247.
2. Each embedding of a closed convex curve has dilation ≥ 1.00157.
3. Let P be the plane graph that results from intersecting n infinite families of equidistant, parallel lines in general position. Then the vertex set of P has dilation $\geq 2/\sqrt{3} \approx 1.1547$.

Keywords: Dilation; geometric network; lower bound; plane graph; spanning ratio; stretch factor.

1. Introduction

Transportation networks like railway systems can be modeled by geometric graphs: stations correspond to vertices, and the tracks between stations are represented by arcs. One measure of the performance of such a network P is given by its *vertex-to-vertex dilation*. For any two vertices, p and q, let $\pi(p, q)$ be a shortest path from

p to q in P. Then,

$$\delta_P(p,q) := \frac{|\pi(p,q)|}{|pq|}$$

measures the detour one encounters in using P, in order to get from p to q, instead of travelling straight; here $|.|$ denotes the Euclidean length. The dilation of P is given by

$$\delta(P) := \sup_{p,q \text{ vertices of } P} \delta_P(p,q).$$

1.1. *Problem statement*

Suppose we are given a set of stations, and we want to build a network connecting them whose dilation is as low as possible. In this work we are assuming that bridges cannot be used. That is, where two or more tracks cross each other, a station is required, that must also be considered in evaluating the dilation of the network.

More precisely, we are given a set S of points in the plane, and we are interested in plane graphs $P = (V, E)$ whose vertex set V contains S, such that the dilation $\delta(P)$ is as small as possible. At this point we are not concerned with the algorithmic cost of computing P, nor with its building cost in terms of the total length of all edges in E, or the size of V—only the dilation of P matters. However, to rule out degenerate solutions like the complete graph over all points in the real plane, we require that the vertex set V of P contains only a finite number of vertices in addition to S. This leads to the following definition.

Definition 1. Let S be a set of points in the plane. Then the *dilation of* S is given by

$$\Delta(S) = \inf\{\ \delta(P); P = (V,E) \text{ plane graph } \& \ S \subseteq V \ \& \ V \setminus S \text{ finite}\ \}.$$

The challenge is in computing the dilation of a given point set. Even for a set as simple as S_5, the vertices of a regular 5-gon, $\Delta(S_5)$ is not known. It is not even clear if $\Delta(S_5) > 1$ holds.[a] Figure 1 depicts some attempts to find good embeddings for S_5.

1.2. *Related work*

In the context of spanners, the dilation is often called the stretch factor or the spanning ratio of a graph P; see Eppstein's handbook chapter, Ref. 11, Arikati et al., Ref. 3, or the forthcoming monograph by Narasimhan and Smid, Ref. 19. However, spanners are usually allowed to contain edge crossings, unlike the plane graphs considered here.

[a]While S_5 is not contained in the vertex set of any plane graph of dilation 1, according to the characterization of dilation-free graphs given by D. Eppstein, Ref. 12, there could be a sequence of plane graphs, each containing S_5, whose dilation shrinks towards 1.

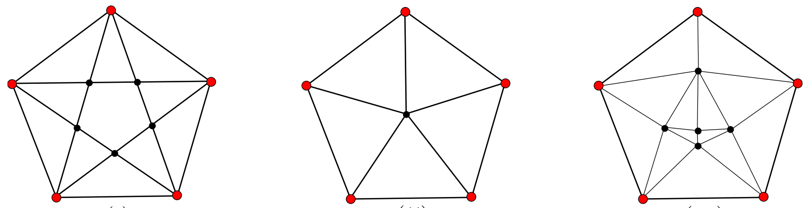

Fig. 1. Embedding five points in regular position into the vertex set of a plane graph of low dilation. (i) Constructing the complete graph results in a new 5-gon. (ii) A star yields dilation ≈ 1.05146. (iii) Using a 4-gon around an off-center point gives dilation ≈ 1.02046, as was shown by Lorenz, Ref. 18.

Substantial work has been done on proving upper bounds to the dilation of certain plane graphs. For example, Dobkin et al., Ref. 5, and Keil and Gutwin, Ref. 16, have shown that the Delaunay triangulation of a finite point set has a dilation bounded from above by a small constant. The best upper bound known is 2.42, but a better bound of $\pi/2$ is conjectured to hold. Moreover, there are structural properties of plane graphs, like the good polygon and diamond properties, which imply that the dilation is bounded from above, see Das and Joseph, Ref. 4. This result implies that the minimum weight and the greedy triangulations also have a dilation bounded by a constant. Our approach differs from this work, in that the use of extra vertices is allowed. This will lead to an upper bound considerably smaller than $\pi/2$.

Quite recently, a related measure called geometric dilation has been introduced, see Ref. 10, 8, 1, 6, 9, and 7, where *all* points of the graph, vertices and interior edge points alike, are considered. The small difference in definition leads to rather different results which do not apply here. For example, plane graphs of minimum geometric dilation tend to have curved edges, whereas for the vertex-to-vertex dilation, straight edges work best.

1.3. *New results*

In this paper we provide the first lower and upper bounds to the dilation of point sets, as defined in Definition 1. First, in Section 2, we prove a structural property similar in spirit to the good polygon and diamond properties mentioned above. If a plane graph P contains a face R whose diameter is a weak local maximum, so that each face in a certain neighborhood of R has a diameter at most a few percents larger than R, then the dilation of P can be bounded away from 1. We will derive the following consequence. If C denotes a closed convex curve then $\Delta(C) > 1.00157$ holds for the set of points on C, i. e., each point on curve C is considered a degree 2 vertex; see Figure 2 (i). Another consequence: If P is a plane graph whose faces have diameters bounded from above by some constant then $\delta(P) > 1.00156$ holds.

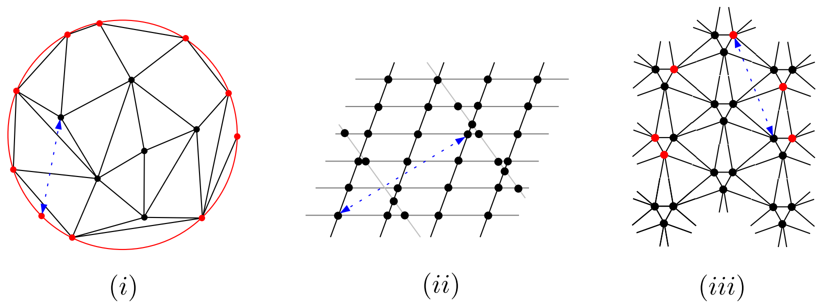

Fig. 2. Results. (i) Each embedding of a closed convex curve has dilation > 1.00157. (ii) Each such arrangement has dilation > 1.1547. (iii) Each finite point set has dilation < 1.1247.

When looking for plane graphs of low dilation that can accommodate a set of given points, grids come to mind. Even the simple quadratic grid consisting of equidistant vertical and horizontal lines has a vertex-to-vertex dilation of only $\sqrt{2} \approx 1.414$, and we can force its vertex set to contain any finite number of points with rational coordinates, by choosing the cell size appropriately. How to accommodate points with real coordinates is discussed in Section 4. If we use three families of lines, as in the tiling of the plane by equilateral triangles, a smaller dilation of only $2/\sqrt{3} \approx 1.1547$ can be achieved; see Figure 3.

An interesting question is if the dilation can be decreased even further by using lines of more than three different slopes. The answer is somewhat surprising, because parallel highways, a mile apart, for each orientation $2\pi i/n$, would in fact provide very low dilation to long distance traffic. But there are always vertices relatively close to each other, for which the dilation is at least $2/\sqrt{3}$ as we shall prove in Section 3.

Yet in Section 4 we introduce a way of getting below the $2/\sqrt{3}$ bound offered by the equitriangular tiling depicted in Figure 3. One can modify this tiling by replacing each vertex with a triangle, and by connecting neighboring triangles as shown in Figure 2 (iii). The resulting graph has a dilation less than 1.1247. We can scale, and slightly deform this graph, so that its vertex set contains any given finite set of points; then we cut off the unbounded part which does not host any point. These operations increase the dilation by some factor that can be made arbitrarily small. Thus we obtain that $\Delta(S) < 1.1247$ holds, for every finite point set S.

Finally, in Section 5, we address some of the questions left open and discuss future work.

2. A Lower Bound

First, we introduce some notations. Let P be a plane graph, and let R be a face of P with boundary $\partial(R)$. As usual, let

$$\mathrm{diam}(R) = \sup\{\ |ef|\ ;\ e, f \text{ vertices of } R\}$$

be the *diameter* of R; unbounded faces have infinite diameter. Now let R be a bounded face of P. For any positive number r, the *r-neighborhood of face* R is defined as the set of all faces of P that have non-empty intersection with a disk of radius r centered at the midpoint of a segment ef, where e, f are vertices on $\partial(R)$ satisfying $|ef| = \mathrm{diam}(R)$; see Figure 4. If there are more than one pair e, f of vertices of this kind we break ties arbitrarily. One should observe that the r-neighborhood of a bounded face may include unbounded faces of P.

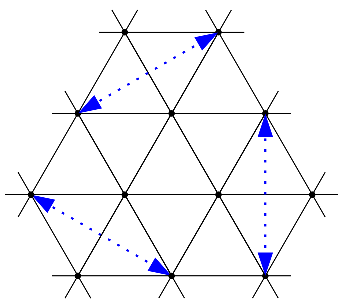

Fig. 3. The tiling by equilateral triangles is of dilation $2/\sqrt{3} \approx 1.1547$.

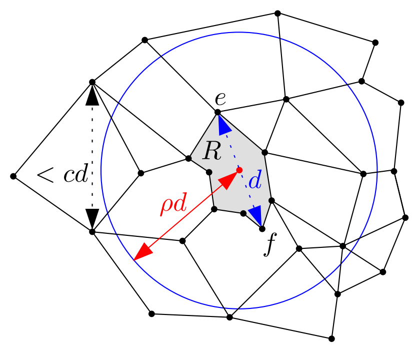

Fig. 4. No face intersected by the disk is of diameter $> cd$.

The results of this section are based on the following lemma.

Lemma 1. *For each parameter $c \in [1, 1.5)$ there exist numbers $\rho > 1$ and $\delta > 1$ such that the following holds. Suppose that the plane graph P contains a bounded face R of diameter d, such that all faces in the ρd-neighborhood of R have diameter less than cd.*[b] *Then the dilation of P is at least δ.*

Proof. We may assume that face R is of diameter $d = 1$. Also, we assume that ef is vertical and that its midpoint equals the origin; see Figure 5. As no vertex of $\partial(R)$ has a distance > 1 from e or from f, face R is completely contained in the lune spanned by e and f.[c]

[b]This implies that only bounded faces can be included in the ρd-neighborhood of R.
[c]The lune spanned by e and f equals the intersection of the two circles of radius $|ef|$ centered at e and f, respectively.

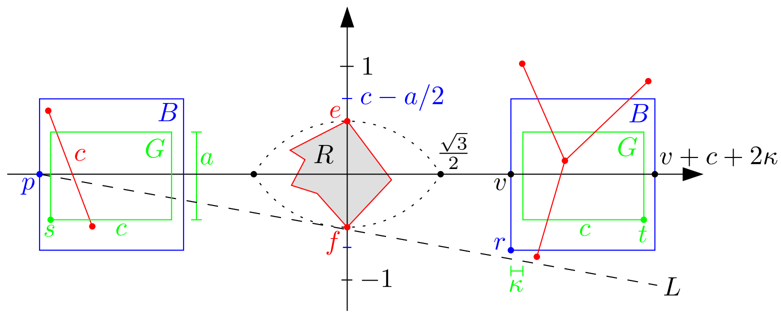

Fig. 5. Constructing a lower bound.

Now we place two axis-parallel boxes of width c and height a symmetrically on the X-axis, at a distance of $v+\kappa$ to either side of the origin; these boxes are denoted by G in Figure 5. The parameters a, v, and κ will be chosen later in such a way that $a < 1$ and $c \geq 1$ hold. Suppose that all faces of P that intersect the disk of radius $\rho := v + c + \kappa$ about the origin have a diameter less than c.

Now, let us consider such a box, G. As its width equals c, there cannot exist two points on the left and on the right vertical side of G, respectively, that are contained in the same face R' of the graph. Namely, the point on the side closer to the origin would have a distance of at most $\sqrt{(v+\kappa)^2 + \frac{a^2}{4}} \leq \sqrt{(v+\kappa)^2 + \frac{1}{4}} < \rho$ from the origin, so that face R' would intersect the disk of radius ρ, which would imply $\mathrm{diam}(R') \geq c$. Thus, the vertical sides of G must be separated by P, i. e., there must be a sequence of edges, or a single edge, cutting through the upper and lower horizontal sides of G. In the first case, box G must contain a vertex of P, as shown on the right hand side in the figure. In the second case, the edge that crosses G top-down must itself be of length $< c$, because it belongs to a face intersected by the disk of radius ρ. We enclose G in the smallest axis-parallel box B which contains all line segments of length c that cross both horizontal sides of G. The outer box B is of height $2c - a$, its width exceeds the width c of G by $\kappa = \kappa(a, c)$ on either side, to include all slanted segments. We analyze the function $\kappa(a, c)$ in Lemma 2 after finishing this proof. By construction, both the upper and the lower half of B must contain a vertex of P in the second case, as shown on the left hand side of Figure 5.

Now we discuss how to choose the parameters v and a such as to guarantee a dilation of $\delta > 1$ in either possible case. First, we let $v > \sqrt{3}/2$ so that the boxes B are disjoint from the lune; consequently, every shortest path in P connecting vertices in the two boxes B has to go around the face R.

Case 1. Each of the boxes G contains a vertex of P. If a is less than $|ef| = 1$ then these vertices cause a dilation of at least a certain value $\delta > 1$ which depends only on a, c, v and κ.

Case 2. Each of the boxes B contains two vertices of P, one above, and one below the X-axis. Let p and r denote vertices in the upper part of the left and in the lower part of the right box B, respectively. We assume w. l. o. g. that the shortest path in P connecting them runs below vertex f, so that its length is at least $|pf|+|fr|$. If we make sure that, even at the extreme position depicted in Figure 5, vertex r lies above the line L through p and f, a dilation $\delta > 1$ is guaranteed. The equation of L is given by

$$Y = -\frac{1}{2(v+c+2\kappa)}\, X - \frac{1}{2}.$$

Thus, we must ensure that

$$-(c - \frac{a}{2}) > -\frac{1}{2(v+c+2\kappa)}\, v - \frac{1}{2}$$

or, equivalently,

$$a > 2c - 1 - \frac{v}{v+c+2\kappa}$$

holds. Together with the condition $1 > a$ from Case 1 we obtain

$$3 - 2c > a + 2 - 2c > \frac{c+2\kappa}{v+c+2\kappa} > 0. \qquad (*)$$

Case 3. There exist at least one vertex of P in the left box G, and at least two vertices in the right box B, above and below the X-axis. Since the vertex in the box G must reside in either the upper or the lower part of the enclosing box B, Case 2 applies.

Clearly, the above conditions can be fulfilled for each given $c \in [1, 1.5)$. First, we pick $a \in (2c-2, 1)$, which guarantees $3 - 2c > a + 2 - 2c > 0$. Then we choose $v > \sqrt{3}/2$ so large that the second inequality in condition (*) is satisfied. This proves Lemma 1. □

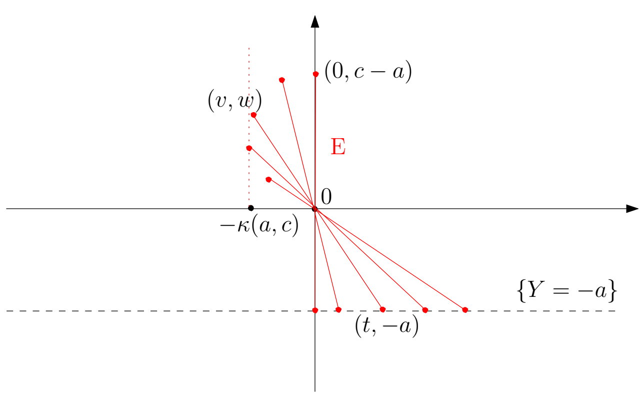

Fig. 6. Analyzing the motion of the left endpoint of the line segment.

Now, we analyze the function $\kappa(a, c)$ that occurred in the proof above, see Figure 6 for an illustration.

Lemma 2. *Let E be a line segment of length c that passes through the origin and has its lower endpoint on the horizontal line $\{Y = -a\}$, where $0 < a < c$. Then its upper endpoint's distance to the Y-axis is at most*

$$\kappa(a, c) := a \left((\frac{c}{a})^{\frac{2}{3}} - 1 \right)^{\frac{3}{2}} .$$

Proof. Let (v, w) and $(-a, t)$ denote the Cartesian coordinates of the upper and lower endpoints of E, correspondingly. Since (v, w) is located on the line through the origin and $(t, -a)$, we have $w = -\frac{a}{t} v$. Moreover, the length of E is given by $c^2 = (v - t)^2 + (a + w)^2$. This yields a quadratic equation for v that solves to

$$v = t - \sqrt{\frac{c^2 t^2}{t^2 + a^2}}$$

for $t < 0$. By taking the derivative of $v = v(t)$ we find a unique maximum at $t = \sqrt{c^{\frac{2}{3}} a^{\frac{4}{3}} - a^2}$. Plugging this value into the equation for v leads to the desired result. □

It is quite straightforward to derive quantitative results from the construction in the proof of Lemma 1, by adjusting the values of the parameters a and v. The following numerical values for ρ and δ have been obtained using Maple.

c	1.0	1.001	1.1	1.2	1.3	1.4
δ	1.00157	1.00156	1.00043	1.000092	1.000012	1.00000056
ρ	1.923	1.925	2.46	3.9	6.9	16.5

As a first consequence, we get the following result.

Theorem 1. *Let P be an infinite graph whose faces cover the whole plane and have a diameter bounded from above by some constant. Then $\delta(P) > 1.00156$ holds for its dilation.*

Proof. By assumption, $d^* := \sup\{\text{diam}(R) : R \text{ face of } P\}$ is finite. For each $\epsilon > 0$ there exists a face R of P such that $\text{diam}(R) > (1 - \epsilon) d^*$ holds. By the assumption on graph P, *all* faces R' of P, in particular those in any neighborhood of R, satisfy

$$\text{diam}(R') \le d^* < \frac{1}{1 - \epsilon} \text{ diam}(R) \le 1.001 \text{ diam}(R),$$

if ϵ is small enough. Thus, graph P has dilation at least 1.00156. □

The second consequence of Lemma 1 is a lower bound to the Δ function.

Theorem 2. *Let C denote the set of points on a closed convex curve. Then $\Delta(C) > 1.00157$ holds for its dilation.*

Proof. Suppose one could achieve a dilation $\delta \leq 1.00157$ by adding to curve C a finite plane graph, Q. We may assume that Q is fully encircled by C, since edges outside of C would not provide shortcuts between points on C, by convexity. Let R be a face of Q of maximum diameter $d = |ef|$. As in the proof of Lemma 1 we assume $|ef| = 1$, $e = (0, \frac{1}{2})$, $f = (0, -\frac{1}{2})$. We apply the same construction as there, using the parameters $c = 1.0$ and $\rho = 1.923$; see Figure 5.

Suppose that one of the two boxes, G, is completely outside of C. A quick calculation[d] shows that the other box, G', is either intersected by C, so that it contains a vertex, or it is completely encircled by C, and we can argue as before. In either case, its enclosing box B' must contain a vertex r, w.l.o.g. in its lower part.

Now let p be an arbitrary point in the upper part of B, the enclosing box of G. We know that the shortest Euclidean path from p to r around the barrier ef would have dilation $\delta(p, q) > 1.00157$. As we walk straight from p in direction of r the dilation increases, since the Euclidean distance from our current position, p', to r decreases at least as fast as the path length $|p'e| + |er|$ does. Before p' reaches the vertical line through ef, where the dilation would be no longer defined, it must hit a point of C. Thus, we have obtained two vertices of $C \cup Q$ that have dilation > 1.00157— a contradiction. If both boxes G are at least partly inside of C, we can apply the arguments from case 1 in the proof of Lemma 1. $\square$

For the circle one can find an embedding of dilation $(1 + \epsilon)/\sin 1 \approx 1.188$ by placing a single vertex at the center and adding many equidistant radial segments.

3. A Lower Bound for Line Arrangements

A lower bound much stronger than 1.00157 can be shown for graphs that result from intersecting n families F_i of infinitely many equidistant parallel lines. Each family is defined by three parameters, its orientation α_i, the distance w_i in X-direction between consecutive lines, and the offset distance e_i from the origin to the first line in positive X-direction. We say that such families are in *general position* if the numbers w_i^{-1} are linearly independent over the rationals[e].

Theorem 3. *Given n families F_i, $2 \leq i \leq n$, each consisting of infinitely many equidistant parallel lines. Suppose that these families are in general position. Then their intersection graph P is of dilation at least $2/\sqrt{3}$.*

One should observe that this lower bound is attained by the equitriangular grid shown in Figure 3.

[d]Let c_1, c_2 denote the two points where C intersects the X-axis. By assumption, $\delta(c_1, c_2) \leq \delta \leq 1.00157$. Since the detour between c_1, c_2 enforced by ef is a minimum if ef bisects c_1c_2, $|c_1c_2| > 1/\sqrt{\delta^2 - 1} \geq 17.8$ holds, by Pythagoras' theorem. On the other hand, the distance between the two boxes G is only $2v < 2(\rho - c) = 1.846$.

[e]This means, if $\sum_{i=1}^{n} a_i \; w_i^{-1} = 0$ holds for rational coefficients a_i then each a_i must be zero.

Proof. For $n \leq 3$ the claim can be proven quite easily without assuming general position. For three families of infinitely many, equidistant parallel lines their arrangement has dilation $\geq 2/\sqrt{3}$, if at least two families have different orientations.

Let β be the largest angle between consecutive orientations, and let R_2 be the sub-lattice formed by these two families. Clearly, $\beta \geq 60°$. For suitable integers a, b, $a \times b$ cells of R_2 will form a parallelogram, Q, two of whose internal angles are of value β and whose sides are of identical length, up to an error that can be made arbitrarily small. Since the third family cannot provide a shortcut between the vertices of Q with interior angle β, a shortest connecting path in R must follow the boundary of Q, i. e., it consists of two edges of the same length that meet at the angle $\pi - \beta \leq 120°$. Consequently, its dilation is at least $1/\sin(120°/2) = 2/\sqrt{3}$; see Figure 2 (ii).

Now let $n > 3$. We shall prove the existence of a face R of P that represents so large a barrier between two vertices p and p' of P that even the Euclidean shortest path from p to p' around R is of dilation at least $2/\sqrt{3}$. In fact, we shall provide such a face and two vertices that are *symmetric* about the same center point, which greatly helps with our analysis; see Figure 7. To this end we use the general position assumption, and apply Kronecker's theorem on simultaneous approximation in its following form, see Ref. 2.

Theorem 4. *(Kronecker) Let L be a line in $\mathbb{R}^n$ that passes through the origin and through some point $(y_1, \ldots, y_n)$ whose coordinates are linearly independent over the rationals. For each point $t \in \mathbb{R}^n$, and for each $\epsilon > 0$, there is an integer translate $t + m$, $m \in \mathbb{Z}^n$, of t whose $\epsilon-$neighborhood is visited by L.*

In other words, line L is dense on the torus $\mathbb{R}^n/\mathbb{Z}^n$. Proofs can be found in, e. g., Apostol, Ref. 2, or Hlawka, Ref. 15.

Since the families of lines are in general position, the real numbers $y_i := w_i^{-1}, 1 \leq i \leq n$, do not satisfy a linear equation with rational coefficients. Kronecker's theorem, applied to $t_i := e_i/w_i + 1/2$ and $\epsilon > 0$ yields the existence of integers m_i and of a real number x satisfying

$$|\frac{e_i}{w_i} + \frac{1}{2} + m_i - x\ \frac{1}{w_i}| < \epsilon$$

that is,

$$|e_i + \frac{1}{2}\ w_i + m_i\ w_i - x| < \epsilon\ w_i.$$

Consequently, the point $(x, 0)$ lies, for each family F_i, halfway between two neighboring lines, so that it is center of symmetry for some face R in P—up to an error that can be made arbitrarily small since the numbers w_i are fixed.

From now on, we assume that R is symmetric about the origin, and that its longest diagonal, d, is vertical and of length 2. Let us assume that the dilation of graph P is less than $2/\sqrt{3}$. We shall derive a contradiction by proving that there exist two families of lines that contribute to the boundary of R and have symmetric intersection vertices p, p' that cause a dilation $> 2/\sqrt{3}$ in the presence of the barrier R. Since R is symmetric, it is sufficient to provide *one* such vertex p satisfying

$$\frac{|pa| + |pb|}{2|p|} \geq 2/\sqrt{3}$$

where a and b are the endpoints of diagonal d, and $|p| = |p0|$ denotes the distance from p to the origin, as shown in Figure 7.

To this end, consider the locus E of all points where equality holds in the above inequality, see Figure 8. E satisfies the quartic equation $(X^2+Y^2-\frac{3}{2})^2 = \frac{9}{4}(1-Y^2)$. We want to show that the right part of its interior—referred to as an *ear*, due to its shape—contains a vertex p of P.

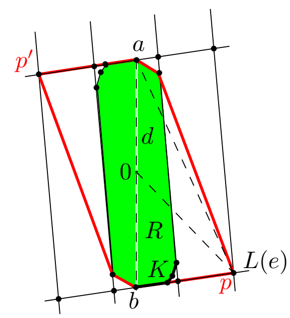

Fig. 7. A symmetric face acting as barrier.

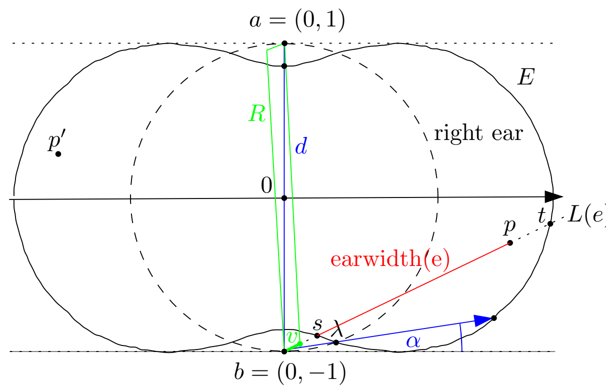

Fig. 8. The locus E of all points that cause dilation $2/\sqrt{3}$ in the presence of line segment ab. The halfline $L(e)$ extends the edge $e = bv$ to the right.

Since d is the longest diagonal of the symmetric face R, the vertices of R are contained in the circle spanned by d. On the other hand, R itself is of dilation $< 2/\sqrt{3}$, by assumption. Hence, the vertices of R must be outside of the locus curve. This leaves only the small caps at a, b for the remaining vertices of R, and $\partial(R)$ must contain two long edges.

First, assume that R is a parallelogram, as shown in Figure 8. Its shape is determined by the position of its lower right vertex v in the bottom cap. Consider the edge e from vertex b to v. Its extension beyond v, $L(e)$, intersects the right ear in a segment of length earwidth(e). In Figure 8 we have earwidth$(e) = |st|$, while the length of the intersection of $L(e)$ with the lower cap equals $|bs|$. The ratio $|st|/|bs|$ takes on its minimum value $3.11566\ldots > 1$ at the angle $\alpha \approx 9.74°$, when $L(e)$ hits the intersection point, λ, of the locus curve with the circle. Because of

$$\text{earwidth}(e) \;>\; 3.11|bs| \;\geq\; 3.11|bv| \;>\; |bv|$$

the segment of $L(e)$ passing through the ear must contain a vertex p of the two line families bounding R. This proves Theorem 3 in case face R is a parallelogram.

It is interesting to observe that in the limiting case $\alpha = 0$, when R degenerates into its diagonal d, the largest possible dilation $2/\sqrt{3}$ is only attainable by picking p as the bottommost point of the ear. This shows why these arguments would not work for any lower bound larger than $2/\sqrt{3}$.

Now let R be a general symmetric convex polygon. We may assume that its two long edges have non-positive slope. Let K denote the convex boundary chain of R that starts at vertex b and leads to the right until it hits the rightmost long edge of R. In this situation the following holds.

Lemma 3. *There exists an edge e in chain K such that the line $L(e)$ passing through e has the following property. The intersection of $L(e)$ with the the right ear of the locus curve is at least twice as long as the vertical projection of chain K onto $L(e)$.*

Because it requires some technical effort, we prove Lemma 3 in Appendix A. Here, we apply it to finish the current proof of Theorem 3. Consider Figure 9. Let $\mathrm{cut}(e)$ denote the length of the segment of $L(e)$ that is cut out by the extensions of the long edges of face R, and let $\mathrm{proj}_K(e)$ be the length of the vertical projection of chain K onto $L(e)$. By translating $L(e)$ to the endpoint of K, we can see $\mathrm{cut}(e) \leq 2\ \mathrm{proj}_K(e)$, so Lemma 3 implies

$$\mathrm{cut}(e) \ \leq\ 2\ \mathrm{proj}_K(e) \ \leq\ \mathrm{earwidth}(e).$$

This guarantees the existence of a vertex of P in the interior of the locus curve and completes the proof of Theorem 3 in the general case. $\square$

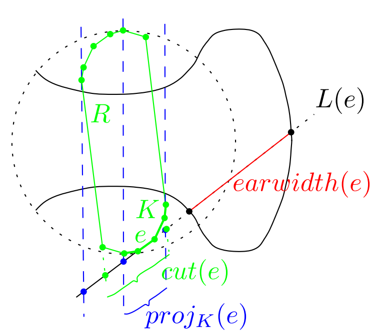

Fig. 9. Vertically projecting the convex chain K onto the line $L(e)$. The intersection of $L(e)$ with the right ear of the locus curve is of length earwidth(e).

4. An Upper Bound to the Dilation of Finite Point Sets

In this section we first define a grid $G(a_{\text{opt}})$ whose dilation $\delta(G(a_{\text{opt}})) \approx 1.124667$ is significantly better than the dilation of the grid of equilateral triangles, $\delta(G_\triangle) = 2/\sqrt{3} \approx 1.1547$, which was shown in Figure 3.

Next, we show how a slightly perturbed version of $G(a_{\text{opt}})$ can be used to embed any finite point set S, thereby proving $\Delta(S) < 1.1247$.

4.1. *Definition of the graph* $G(a_{\text{opt}})$

First, we show how to modify the equitriangular grid, $G_\triangle$, displayed in Figure 3, in order to decrease its dilation. The construction is shown in Figure 10.

Let the origin be a vertex of the original grid $G_\triangle$ of equilateral triangles, let their side length equal one, and let one of the three directions of the appearing edges be parallel to the x-axis. By v_α we denote the vector $v_\alpha := (\cos\alpha, \sin\alpha)$. Then, we can define the vertex set $V_\triangle$ of $G_\triangle$ more formally by

$$V_\triangle := \{iv_{0^\circ} + jv_{60^\circ} \mid i, j \in \mathbb{Z}\} = \left\{ \left(i + \frac{1}{2}j, \frac{\sqrt{3}}{2}j \right) \,\middle|\, i, j \in \mathbb{Z} \right\}$$

Let a be a real number in the interval $(0, 2/3)$. Each vertex of the original triangular

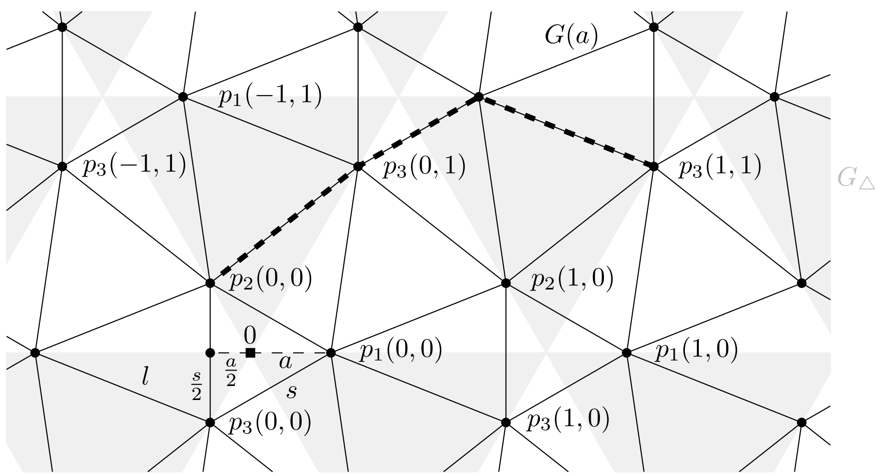

Fig. 10. The graph $G(a)$ and a shortest path connecting a critical pair.

grid is now replaced by an equilateral triangle of side length

$$s(a) := \sqrt{3}a. \tag{1}$$

We put the centers of these triangles, where all the heights intersect, directly on the corresponding vertex of $G_\triangle$. The parameter a equals the distance between a vertex of such a triangle and its center. We assume that one corner of each triangle points towards the positive x-direction.

It is useful to define the vectors pointing from the center of a small triangle to its three vertices.

$$w_1 := av_{0^\circ} = (a, 0)\,, \quad w_2 := av_{120^\circ} = \left(-\frac{a}{2}, +\frac{s}{2}\right), \quad w_3 := av_{240^\circ} = \left(-\frac{a}{2}, -\frac{s}{2}\right).$$

We also take advantage of the equivalent definition

$$w_k = a\Big(\cos\big((k-1)120^\circ\big), \sin\big((k-1)120^\circ\big)\Big) \text{ for } k \in \{1,2,3\}. \tag{2}$$

Then, we can enumerate the vertex set $V(a)$ of the graph $G(a)$ by defining

$$p_k(i,j) := iv_{0^\circ} + jv_{60^\circ} + w_k, \quad V(a) := \{p_k(i,j) \mid i,j \in \mathbb{Z}, k \in \{1,2,3\}\}. \tag{3}$$

The edges are

$$\begin{aligned} E(a) := & \{ (p_k(i,j), p_l(i,j)) \mid i,j \in \mathbb{Z}, k,l \in \{1,2,3\}, k \neq l \} \\ & \cup \{ (p_1(i,j), p_2(i+1,j)) \mid i,j \in \mathbb{Z} \} \cup \{ (p_1(i,j), p_3(i+1,j)) \mid i,j \in \mathbb{Z} \} \\ & \cup \{ (p_1(i,j), p_3(i,j+1)) \mid i,j \in \mathbb{Z} \} \cup \{ (p_2(i,j), p_3(i,j+1)) \mid i,j \in \mathbb{Z} \} \\ & \cup \{ (p_2(i,j), p_1(i-1,j+1)) \mid i,j \in \mathbb{Z} \} \cup \{ (p_2(i,j), p_3(i-1,j+1)) \mid i,j \in \mathbb{Z} \} \end{aligned}$$

The first set of the union contains the edges of the small triangles, whose lengths equal $s(a)$. All the other edges have the length

$$l(a) = \sqrt{\left(1 - \frac{3}{2}a\right)^2 + \left(\frac{s(a)}{2}\right)^2} \overset{(1)}{=} \sqrt{3a^2 - 3a + 1}.$$

It is interesting that the graph $G(a)$ can be seen not only as a modification of the grid of equilateral triangles where each vertex is replaced by a small triangle, but also as a distortion of the triangular grid, because $G(a)$ is graph-theoretically isomorph to $G_\triangle$.

In Appendix B we prove that the dilation $\delta(G(a))$ attains its minimum 1.1246665... for a value $a_{\text{opt}} = 0.2485694...$. Hence, we try to use the graph $G(a_{\text{opt}})$ to embed any given finite point set.

4.2. *Embedding arbitrary finite point sets*

We obtain the following consequence.

Theorem 5. *Each finite point set S is of dilation $\Delta(S) < 1.1247$.*

Proof. We show how to embed any finite point set, S, into the vertex set of a —slightly deformed— copy of the graph $G(a_{\text{opt}})$ shown in Figure 11. The proof uses a technique introduced in Ref. 8 based on the approximation of reals by rationals.

First, we choose a subset of vertices, U, that form a rectangular sub-grid whose cells have size $\gamma \times \sqrt{3}\gamma$, where γ is a scaling factor for $G(a_{\text{opt}})$; see Figure 11. Only

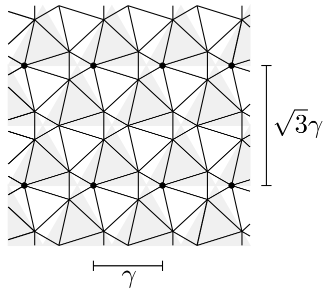

Fig. 11. The vertex subset U of $G(a_{\text{opt}})$ scaled by the factor γ.

the vertices in U will be used for embedding S. Let $S = \{p_i, 1 \leq i \leq n\}$, where $p_i = (\alpha_i, \beta_i)$. Given an error tolerance $\epsilon > 0$, we want to find integers m_i, n_i, and a cell size γ such that

$$|\alpha_i - m_i\gamma| < \epsilon\gamma \quad \text{and} \quad \left|\beta_i - n_i\sqrt{3}\gamma\right| < \epsilon\gamma$$

hold. The existence of such numbers m_i, n_i, and γ could be directly derived from Kronecker's Theorem 4, if we knew about the linear independence of the coordinates. But we can also resort to a weaker result by Dirichlet's; see Ref. 2.

Theorem 6. *(Dirichlet) Let $\gamma_1, \ldots, \gamma_N$ be real numbers. Then there exist infinitely many integers k_i and q that satisfy*

$$\left|\gamma_i - \frac{k_i}{q}\right| < \frac{1}{q^{1+1/n}}$$

We apply Theorem 6 to $(\gamma_i)_{i=1\ldots 2n} = (\alpha_1, \ldots, \alpha_n, \beta_1/\sqrt{3}, \ldots, \beta_n/\sqrt{3})$. We can choose q arbitrarily big and select a value satisfying $q > (\sqrt{3}/\epsilon)^n$. With the result of the Theorem we define $(m_1, \ldots, m_n, n_1, \ldots, n_n) := (k_i)_{i=1\ldots 2n}$ and $\gamma := 1/q$. Now, if we move vertex $(m_i\gamma, n_i\sqrt{3}\gamma)$ of U to point p_i, this deformation of $G(a_{\text{opt}})$ (scaled by γ) is arbitrarily small relative to γ, so that the dilation is affected by an arbitrary small factor only. Finally, we cut off the excess part of $G(a_{\text{opt}})$ along edge sequences each finite subsequence of whom is a shortest path. Consequently, all vertices in the remaining finite part, F, of $G(a_{\text{opt}})$ are connected by a shortest path within F. □

5. Conclusion

We have introduced the notion of the dilation of a set of points, and proven a non-trivial lower bound to the dilation of the points on a closed curve.

The big challenge to prove a similar lower bound for a finite set of points S has been solved after the presentation of this paper. Klein and Kutz showed that the set Q of 200 points placed evenly on the boundary of a square is of dilation $\Delta(S) >$

1.0000047, see Ref. 17. They also proved that there exists a lower bound $\lambda > 1$ to the dilation of any finite point set S which is not the subset of the vertices of a dilation-free graph, see Eppstein, Ref. 12. However, the question remains open if one can find better upper and lower bounds to the dilation of finite point sets like, e.g. S_5, or if one can even calculate the exact value.

As to the arrangements of lines, we conjecture that our lower bound holds without the assumption of general position. Another interesting question is the following. What is the lowest possible dilation of a graph whose faces cover the whole plane and have bounded diameter? Our results place this value into the interval $(1.00157, 1.1247)$. Any progress on the upper bound might lead to an improvement of Theorem 5.

6. Acknowledgement

We would like to thank the anonymous referees for their valuable comments.

Appendix A. Proof of Lemma 3

In this appendix, we prove Lemma 3. Let the chain K be defined as in the proof of Theorem 3 on page 12, cf. Figure 12.

Lemma 3. *There exists an edge e in chain K such that the line $L(e)$ passing through e has the following property. The intersection of $L(e)$ with the the right ear of the locus curve is at least twice as long as the vertical projection of chain K onto $L(e)$.*

Proof. Consider Figure 12. Let λ be the lower point of intersection of the circle with the right ear curve, E. When traversing chain K from vertex b to the right, its edges appear in increasing order of slope. Let f be the first edge whose extension line, $L(f)$ passes above point λ. *We are going to show that edge f or its predecessor, e, fulfils the claim of the lemma.*[f]

First, we consider edge f. Its line $L(f)$ is defined by the equation

$$Y = \cot\alpha\ X - (1 + d),$$

where α is its angle with the Y-axis, and d denotes the difference in Y-coordinates between vertex $b = (0, -1)$ and $L(f)$'s intersection point with the Y-axis; see Figure 12. Since $L(f)$ is required to pass above $\lambda = (1/3, 2\sqrt{2}/3)$, the pair (α, d) must satisfy the conditions

$$\alpha \geq 0 \tag{A.1}$$

$$\cot\alpha \geq 3(1 + d) - 2\sqrt{2}. \tag{A.2}$$

[f]For now we assume that both edges, e and f, exist in K. The special cases where all edges of K are passing below (or above) λ will be discussed at the end of this proof.

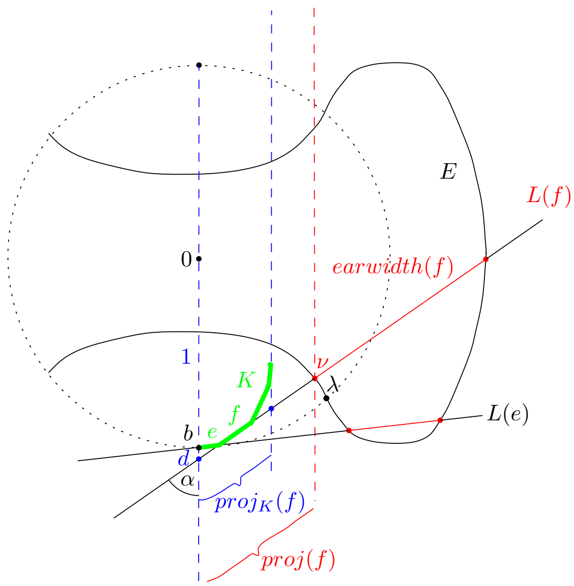

Fig. 12. The first edge f, as seen from vertex b, of the convex chain K whose extension line $L(f)$ runs above point λ.

In particular, angle α is bounded from above by $\cot^{-1}(3-2\sqrt{2}) = 1.4008\ldots$.

The length, $\mathrm{proj}_K(f)$, of the vertical projection of chain K onto line $L(f)$ cannot exceed the length, $\mathrm{proj}(f)$, of $L(f)$ between the Y-axis and the point ν where it enters the ear. Thus, for all parameters (α, d) that satisfy the inequality

$$\frac{\mathrm{earwidth}(f)}{2\ \mathrm{proj}(f)} \geq 1 \tag{A.3}$$

does edge f fulfil Lemma 7 . In order to determine for which (α, d) condition (3) holds we need to compute the intersection points of $L(f)$ with the ear curve. This can be done using the formulae

$$X = \frac{1}{2}\sqrt{-4Y^2 + 6 + 6\sqrt{-Y^2+1}} \tag{A.4}$$

$$X = \frac{1}{2}\sqrt{-4Y^2 + 6 - 6\sqrt{-Y^2+1}} \tag{A.5}$$

that describe the "vertical" part and the upper and lower "horizontal" parts of the right ear, that are separated by the two points of minimum and maximum Y-values.

Figure 13 shows in black the planar surface patch of all points $(\alpha, d, 1)$ where α and d satisfy conditions (1) and (2). In dark grey the ratio $\mathrm{earwidth}(f)/(2\ \mathrm{proj}(f))$ is shown, as a function of α and d. While this ratio stays above 1 for large values of α or small values of d, there are critical pairs $(\alpha, d) \in [0..0.34] \times [0.7..\infty]$ where the value of $2\mathrm{proj}(f)$ is larger than $\mathrm{earwidth}(f)$, so that edge f cannot be used for our proof.

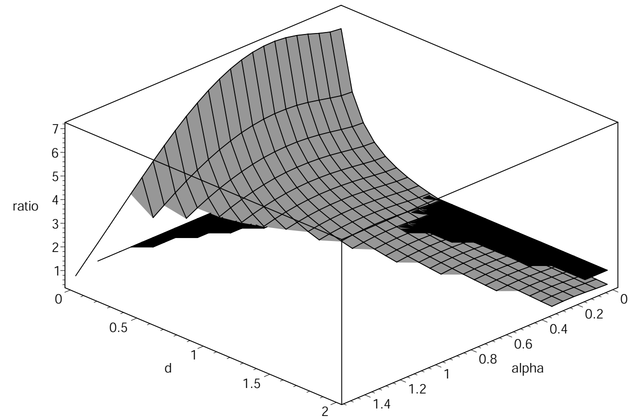

Fig. 13. Not all ratios earwidth$(f)/(2\ \mathrm{proj}(f))$ are bigger than 1.

But if $L(f)$ is steep then the vertical projection of the part of chain K starting with f must be small! Therefore, we shall now look at the predecessor edge, e, of f, that is, the last edge in chain K whose extension line passes below point λ; see Figure 14. We assume that the parameters (α, d) of $L(f)$ are given, and we want to determine if edge e fulfils the Lemma.

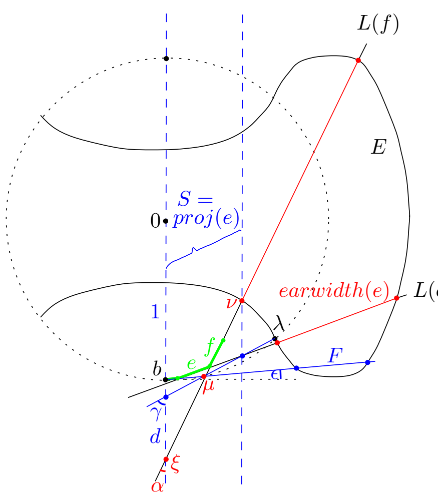

Fig. 14. The last edge e of the convex chain K whose extension line $L(e)$ runs below point λ.

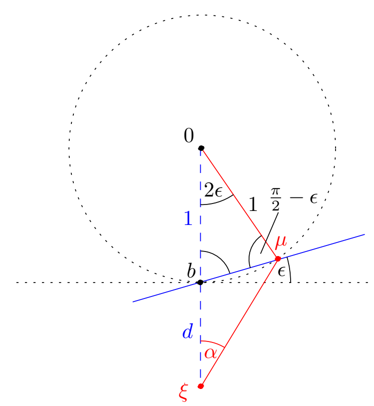

Fig. 15. In the triangle $(0, b, \mu)$ the angles at b and μ are both of value $\pi/2 - \epsilon$.

Let ξ, μ, ν be the intersection points of $L(f)$ with the Y-axis, with the lower part of the circle, and with the lower part of the ear, correspondingly. Moreover, let b be the first vertex of chain K, and let λ denote the lower intersection between circle and ear, as before.

We observe that line $L(e)$ has to pass below the points b and λ, and above point μ. The length of the vertical projection of chain K onto $L(e)$ is bounded from above by $\mathrm{proj}(e)$, the length of $L(e)$ between the two vertical lines through b and ν; these lines do not depend on e. Hence, $\mathrm{proj}(e)$ takes on its largest possible value, S, if $L(e)$ is as steep as possible, that is, if $L(e)$ runs through μ and λ. On the other hand, $\mathrm{earwidth}(e)$ attains its minimum possible value, F, if $L(e)$ is as flat as possible, by passing through b and μ. Therefore, edge e fulfils the Lemma if

$$\frac{F}{2S} \geq 1 \tag{A.6}$$

holds. Clearly, $S = \nu_x / \sin\gamma$, where ν_x denotes the X-coordinate of ν and γ is the angle between $L(\mu, \lambda)$ and the Y-axis given by

$$\gamma = \tan^{-1} \frac{\lambda_x - \mu_x}{\lambda_y - \mu_y} \tag{A.7}$$

In order to compute F we need the angle ϵ between $L(b, \mu)$ and the X-axis. Applying the sine theorem to the triangle spanned by $0, \xi, \mu$ one obtains

$$\frac{1}{\sin\alpha} = \frac{1+d}{\sin(\pi - \alpha - 2\epsilon)}. \tag{A.8}$$

Here $1 + d$ equals the distance between 0 and ξ, and the size of the opposite angle is illustrated by Figure 15. Hence,

$$\epsilon = \frac{\sin^{-1}((1+d)\sin\alpha) - \alpha}{2} \tag{A.9}$$

Given α and d, we are now able to compute the values of F and S in (6). Figure 16 depicts the values of $F/(2S)$ over the critical area $[0..0.34] \times [0.7..\infty]$ where edge f failed to work. Not only is $F/(2S)$ bigger than 1—its value goes even to infinity as α tends to 0.

Figure 17 shows the same three surfaces over the area $[1.2..1.4] \times [0..0.05]$, where only some very narrow patches are visible in Figure 16. Here, edge f can be chosen, while choosing edge e would not work. Indeed, if d is almost equal to 0 then edge e must be very flat, causing its earwidth to be very small, as compared to the projection of edge f onto $L(e)$, which is large because of $\alpha \geq 1.2$.

It remains to discuss the cases where all edges of chain K pass above or below point λ. In the first case we can choose edge f, which is the first edge of chain K in this case, i. e. $d = 0$; see Figure 13. In the second case, choosing edge e, the last edge of K, will work; we can formally add an almost vertical edge f to the chain, and apply the arguments from above.

This concludes the proof of Lemma 3 in the general case. □

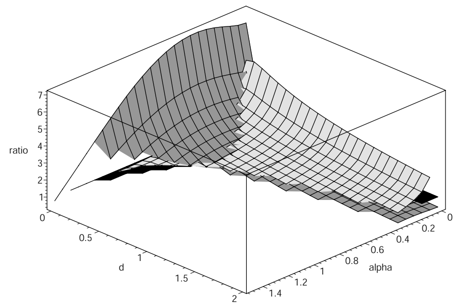

Fig. 16. One of the ratios associated with the edges e and f is always bigger than 1. The surface associated with edge e is shown in light grey, while the surface of f is dark.

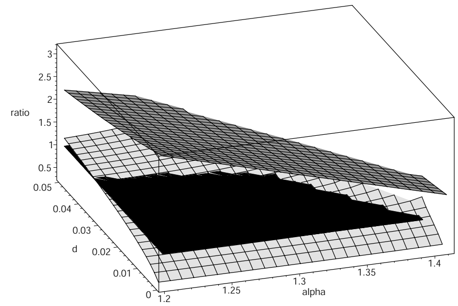

Fig. 17. For large values of angle α, edge f fulfils the Lemma, while e fails.

Appendix B. Optimizing the Dilation of $G(a)$

In this appendix we prove that the dilation of the graph $G(a)$ defined in Section 4.1 attains its minimum value 1.1246665... for a parameter $a_{\mathrm{opt}} = 0.2485694...$.

To this end, we first prove that the dilation of a critical pair attains its minimum for $a = a_{\mathrm{opt}}$. Then, we show that the dilation of $G(a_{\mathrm{opt}})$ is indeed determined by the critical pairs.

B.1. *Dilation of a critical pair*

In this section we will calculate the value a_{opt} which minimizes the dilation of a pair of vertices we call critical pair. A pair of vertices $(p, q) \in V(a)$ is a *critical pair* if there exists a rigid motion[g] $M : \mathbb{R}^2 \to \mathbb{R}^2$ such that $M(G(a)) = G(a)$, $M(p) = p_2(0,0)$ and $M(q) = p_3(1,1)$. For the calculation of a_{opt} it suffices to consider the critical pair consisting of $p_2(0,0)$ and $p_3(1,1)$. By the various symmetries of the graph there are of course many critical pairs with the same shortest path distance and the same Euclidean distance. We consider only the case $a \leq 1/3$. In the end of this section we will explain why this restriction is no problem.

It is easy to see that there are only two candidates for a shortest path in $G(a)$ connecting a critical pair. The length is either $2l(a) + s(a)$ as shown in Figure 10 or $3l(a)$. All other paths are strictly longer. Our assumption $a \leq 1/3$ yields $-3a+1 \geq 0$, which implies $3a^2 - 3a + 1 \geq 3a^2$. We get

$$l(a) = \sqrt{3a^2 - 3a + 1} \geq \sqrt{3}a = s(a).$$

Therefore, we know that $2l(a) + s(a) \leq 3l(a)$, and we can give a rather simple

[g] I.e., M consists only of translations, rotations and reflections.

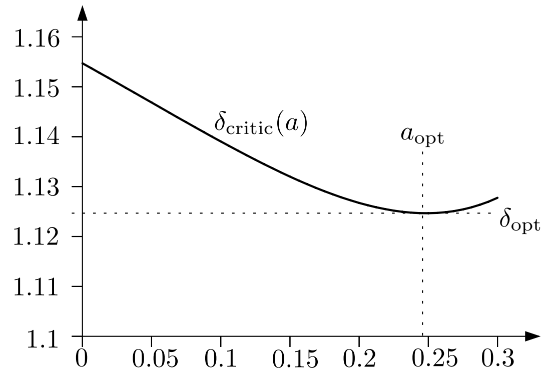

Fig. 18. The dilation δ_{critic} of a critical pair depending on a.

formula for the dilation of a critical pair.

$$\delta_{\text{critic}}(a) := \delta(p_2(0,0), p_3(1,1)) = \frac{2l(a)+s(a)}{\left|\left(\frac{3}{2}-\frac{a}{2}, \frac{\sqrt{3}}{2}-\frac{s(a)}{2}\right)-\left(-\frac{a}{2}, \frac{s(a)}{2}\right)\right|} \quad \text{(B.1)}$$

$$= \frac{2\sqrt{a^2-a+\frac{1}{3}}+a}{\sqrt{a^2-a+1}}.$$

The function is plotted in Figure 18. We want to find the minimum of δ_{critic} for $a \in [0, 1/3]$. If we assume that the occurring denominators do not equal zero, by some additional steps we get

$$\delta'_{\text{critic}}(a) = \frac{\left(\frac{(2a-1)}{\sqrt{a^2-a+\frac{1}{3}}}+1\right)\sqrt{a^2-a+1}-\frac{2a-1}{2\sqrt{a^2-a+1}}\left(2\sqrt{a^2-a+\frac{1}{3}}+a\right)}{(a^2-a+1)}$$

$$= \frac{\frac{2}{3}(2a-1)+\sqrt{a^2-a+\frac{1}{3}}\left(1-\frac{a}{2}\right)}{\sqrt{a^2-a+\frac{1}{3}}\,(a^2-a+1)^{\frac{3}{2}}}$$

The derivative can attain zero only if

$$\frac{2}{3}(2a-1) = \left(\frac{a}{2}-1\right)\sqrt{a^2-a+\frac{1}{3}}$$

$$\Rightarrow \left(a-\frac{2}{3}\right)(9a^3-39a^2-15a+6) = 0$$

This can be seen by squaring the first equation and by simplifying the result. As we are looking for a root of this polynomial in the interval $[0, \frac{1}{3}]$, it must be a root of $9a^3-39a^2-15a+6$. Although this polynomial has three real roots, there is only one root in the considered interval. A numeric approximation yields

$$a_{\text{opt}} = 0.2485694... \quad \text{and} \quad \delta_{\text{critic}}(a_{\text{opt}}) = 1.1246665...$$

We have shown the main arguments to prove the following lemma:

Lemma 4. *Let a_{opt} be the only root of the polynomial $9a^3 - 39a^2 - 15a + 6$ in the interval $[0, \frac{2}{3}]$. Let a be an arbitrary value in $[0, \frac{2}{3}]$. Then we have $\delta(G(a)) \geq \delta_{\text{critic}}(a_{\text{opt}})$.*

Proof. Above we have shown that $\delta_{\text{critic}}(a) \geq \delta_{\text{critic}}(a_{\text{opt}})$ for $a \in [0, \frac{1}{3}]$. This concludes the proof for these a-values, as obviously $\delta(G(a)) \geq \delta_{\text{critic}}(a) \geq \delta_{\text{critic}}(a_{\text{opt}})$.

For the values $a \in [\frac{1}{3}, \frac{2}{3}]$ we consider the points $p_1(0,0)$ and $p_1(1,0)$, cf. Figure 10. The length of the shortest connecting path equals $\min(l(a) + s(a), 3l(a))$ and the Euclidean distance is 1. However, by using the formulas for $s(a)$ and $l(a)$ it is easy to prove that $2l(a) \geq s(a)$. Hence the dilation of this pair equals

$$\delta_G(p_1(0,0), p_1(1,0)) = l(a) + s(a) = \sqrt{3a^2 - 3a + 1} + \sqrt{3}a.$$

By taking the derivative it is not difficult to show that this function is monotonously increasing in $[\frac{1}{3}, \frac{2}{3}]$. Hence, it suffices to show that

$$\sqrt{3a^2 - 3a + 1} + \sqrt{3}a\Big|_{a=\frac{1}{3}} > \delta_{\text{critic}}(a_{\text{opt}}),$$

and this is easy because the left hand side equals $2/\sqrt{3} \approx 1.1547$ which is bigger than $\delta_{\text{critic}}(a_{\text{opt}}) \approx 1.1247$. □

B.2. *Dilation of the optimal graph*

In this section we want to show that the dilation of $G(a_{\text{opt}})$ is attained by a critical pair. Then, we can conclude

$$\forall a \in \left[0, \frac{2}{3}\right]: \quad \delta(G(a)) \overset{\text{Lemma 4}}{\geq} \delta_{\text{critic}}(a_{\text{opt}}) \overset{?}{=} \delta(G(a_{\text{opt}})).$$

As the critical pairs are vertices of the graph $G(a_{\text{opt}})$, the inequality $\delta(G(a_{\text{opt}})) \geq \delta_{\text{critic}}(a_{\text{opt}})$ is trivial. For the other direction we have to show that

$$\forall p, q \in V(a_{\text{opt}}): \quad \delta_{G(a_{\text{opt}})}(p, q) \leq \delta_{\text{critic}}(a_{\text{opt}}) \tag{B.2}$$

First, we will show that it suffices to consider pairs of points whose positions satisfy certain conditions. Because we only consider the graph $G(a_{\text{opt}})$ from now on, we will simplify the notation by using the shorter forms $d(p,q) := d_{G(a_{\text{opt}})}(p,q)$ for shortest path length, $\delta(p,q) := \delta_{G(a_{\text{opt}})}(p,q)$ for dilation, $s := s(a_{\text{opt}})$ for the length of a short edge and $l := l(a_{\text{opt}})$ for the length of a long edge.

Lemma 5. *Let $p, q \in V(a_{\text{opt}})$ be two vertices of the graph $G(a_{\text{opt}})$. Then, there exist two vertices $p', q' \in V(a_{\text{opt}})$ with the same dilation $\delta(p', q') = \delta(p, q)$ which additionally satisfy the following conditions. The point p' is a vertex of the triangle at the origin, and the vector pointing from the origin to the center of the triangle of q' has direction $(\cos\alpha, \sin\alpha)$, $\alpha \in [0°, 30°]$*[h]*.*

[h]From now we will denote this by saying that the triangle of q' lies in direction α.

The same fact can be expressed with the coordinate system by $p' = p_k(0,0)$ *and* $q' = p_{k'}(i', j')$, $0 \le j' \le i'$.

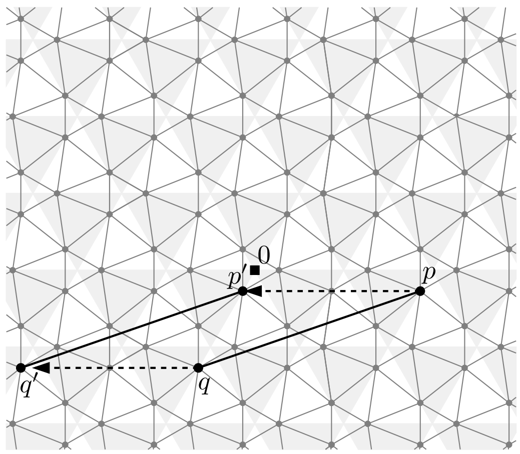

Fig. 19. Without loss of generality we may assume that p belongs to the triangle at the origin.

Proof. Consider Figure 19. Obviously, by translation we can achieve without changing the dilation that p is a vertex of the triangle at the origin.

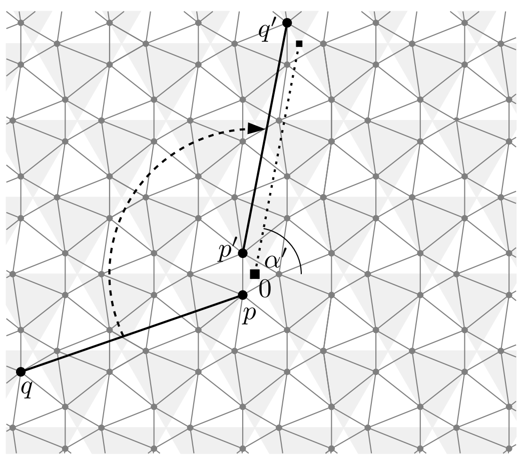

Fig. 20. Without loss of generality we may assume that the triangle of q lies in direction $\alpha \in [0°, 120°]$.

As shown in Figure 20, by rotating the situation by a multiple of 120° we can make sure that the triangle of q lies in a direction satisfying $\alpha \in [0, 120°]$.

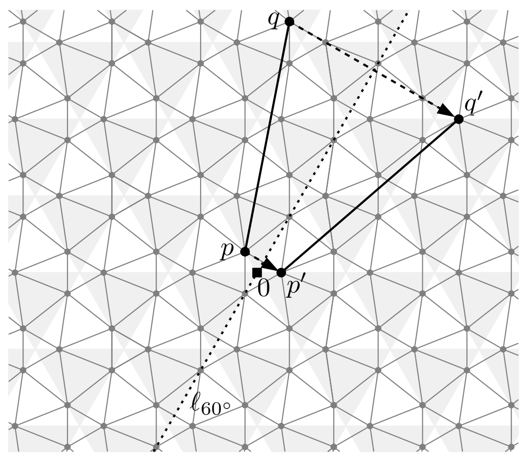

Fig. 21. Without loss of generality we may assume $\alpha \in [0°, 60°]$.

The next step is shown in Figure 21. By possibly reflecting p and q at the line $\ell_{60°}$ in direction $+60°$ through the origin, we can restrict the angle α of the triangle of q to the interval $[0, 60°]$.

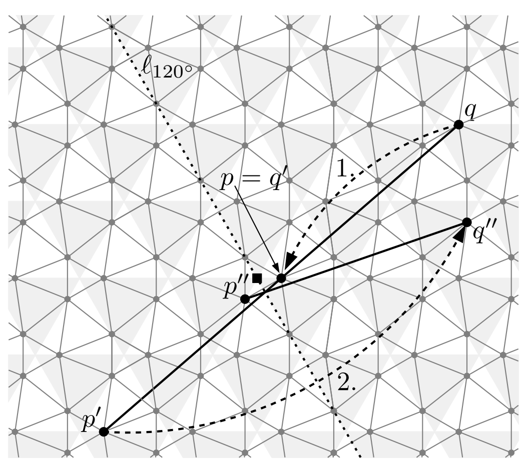

Fig. 22. Without loss of generality we may assume $\alpha \in [0°, 30°]$.

The last step is a little more complicated, see Figure 22. If $\alpha > 30°$ we can translate p and q so that q lies in the triangle at the origin, and then reflect the situation at the line $\ell_{120°}$ in direction $120°$ through the origin. By swapping the roles of p and q, we achieve that p is a vertex of the triangle at the origin, and that $\alpha \in [0°, 30°]$. $\square$

Now we are ready to prove inequality (B.2). Note that for calculating the dilation of a graph, in general, it does not suffice to take into account only vertices which are close to each other. For example, let us consider a rectangular grid. Its dilation equals $\sqrt{2}$. It is attained by vertices whose x-distance equals their y-distance. Such vertices are arbitrarily far away from each other, depending on the least common multiple of the side lengths a and b of the underlying rectangle. If not both side lengths are rational, it can happen that the dilation is not attained by any pair of vertices, but only by a sequence of such pairs where the distance of the two vertices tends to infinity. For instance, this is the case for $a = 1$ and $b = \pi$.

However, as the next step we will show that the detour of vertices of $G(a_{\text{opt}})$ which are far away from each other lies below the detour $\delta_{\text{critic}}(a_{\text{opt}})$ of a critical pair. First, we assume that the two vertices p and q lie in different triangles but on corresponding corners.

Lemma 6. *Let p and q be two vertices of $G(a_{\text{opt}})$ on corresponding vertices of their triangles, i. e. $p = p_k(i, j)$ and $q = p_k(i', j')$. Then, their dilation in the graph $G(a_{\text{opt}})$ is bounded by $\delta(p, q) < 1.101807 =: \delta^{\max}_{\text{same corner}}$.*

Proof. As shown by Lemma 5 we only have to consider pairs of vertices $p, q \in V(a_{\text{opt}})$ where $p = p_k(0, 0)$ and $q = p_{k'}(i', j')$, $0 \le j' \le i'$. Let $\alpha \in [0°, 30°]$ be the direction of the triangle of q. Because p and q lie on corresponding vertices on their triangles, we have $q - p = |pq|(\cos\alpha, \sin\alpha)$.

We want to find a formula for the shortest path distance $d(p, q)$. Each step in the $0°$-direction, the positive x-direction, from a vertex $p_k(i, j)$ to a vertex $p_k(i + 1, j)$ can be done in the graph $G(a_{\text{opt}})$ by using one long edge and one short edge. The Euclidean distance between these vertices equals 1. Hence the dilation in this direction equals

$$\delta^{0°} := \delta(p_k(i, j), p_k(i + 1, j)) = s + l.$$

Each step in the $30°$-direction from a vertex $p_k(i, j)$ to a vertex $p_k(i + 1, j + 1)$ can be done in the graph $G(a_{\text{opt}})$ by using two long edges and one short edge. The Euclidean distance between these vertices equals

$$|p_k(i + 1, j + 1) - p_k(i, j)| = \left|\left(1 + \frac{1}{2}, \frac{\sqrt{3}}{2}\right)\right| = \sqrt{\frac{9}{4} + \frac{3}{4}} = \sqrt{3}.$$

Hence the dilation in this direction is given by

$$\delta^{30°} := \delta(p_k(i, j), p_k(i + 1, j + 1)) = \frac{s + 2l}{\sqrt{3}}.$$

Obviously we get a path from $p = p_k(0, 0)$ to $q = p_k(i', j')$, $0 \le j' \le i'$, by taking $i' - j'$ steps in $0°$-direction and j' steps in $30°$-direction. As displayed in Figure 23,

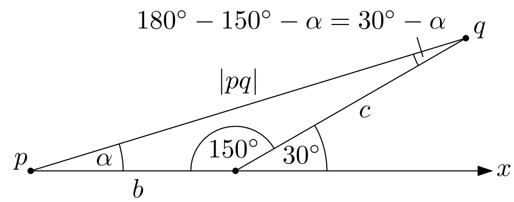

Fig. 23. Subdividing the Euclidean distance $|pq|$ into a horizontal part and a part of direction $30°$.

basic trigonometry yields that the total Euclidean distance b of the $0°$-steps and the total Euclidean distance c of the $30°$ steps equal

$$b = \frac{\sin(30° - \alpha)}{\sin 150°}|pq| = 2|pq|\sin(30° - \alpha), \qquad c = \frac{\sin\alpha}{\sin 150°}|pq| = 2|pq|\sin\alpha.$$

Hence there exists a path in $G(a_{\mathrm{opt}})$ from p to q of length

$$d(p,q) \le b\delta^{0°} + c\delta^{30°} = 2|pq|\left(\sin(30° - \alpha)(s + l) + \sin(\alpha)\frac{s + 2l}{\sqrt{3}}\right).$$

Actually, there holds equality but we do not need to prove this here.

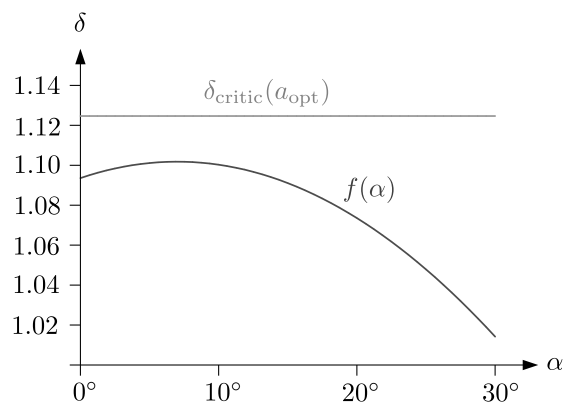

Fig. 24. The dilation of two vertices on corresponding corners depending on the angle α.

The inequality results in the following upper bound on the dilation between the two vertices.

$$\delta(p,q) \le 2\left(\sin(30° - \alpha)(s + l) + \sin(\alpha)\frac{s + 2l}{\sqrt{3}}\right) =: f(\alpha)$$

This function is plotted in Figure 24. Numerical analysis yields the bound stated in the lemma. To be on the save side, we rounded up quite generously. $\square$

The gap between the upper bound on the dilation of vertices on corresponding triangle corners $\delta^{\max}_{\text{same corner}} = 1.101807$ from Lemma 6 and $\delta(G(a_{\mathrm{opt}})) \approx 1.124667$

allows us to show the desired inequality (B.2) for vertices which are far away from each other.

Lemma 7. *Let $p, q \in V(a_{\text{opt}})$ be vertices of the graph $G(a_{\text{opt}})$ whose Euclidean distance satisfies $|pq| \geq 50$. Then, we have $\delta_{G(a_{\text{opt}})}(p, q) < \delta_{\text{critic}}(a_{\text{opt}})$.*

Proof. Let $p = p_k(i, j)$ and $q = p_{k'}(i', j')$ be the considered vertices. If we denote

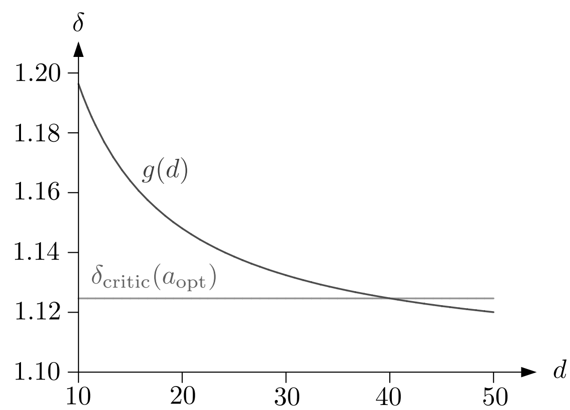

Fig. 25. The upper bound on the dilation between two vertices drops below $\delta_{\text{critic}}(a_{\text{opt}})$ if their Euclidean distance is bigger than ≈ 40.

the distance of p and $p_k(i', j')$ by d, the dilation of p and q is bounded by

$$\begin{aligned}\delta(p,q) &= \frac{d(p_k(i,j), p_{k'}(i',j'))}{|p_k(i,j)p_{k'}(i',j')|} \overset{\triangle-\text{inequ.}}{\leq} \frac{d(p_k(i,j), p_k(i',j')) + d(p_k(i',j'), p_{k'}(i',j'))}{|p_k(i,j)p_k(i',j')| - |p_k(i',j')p_{k'}(i',j')|} \\ &\leq \frac{d(p_k(i,j), p_k(i',j')) + s}{|p_k(i,j)p_k(i',j')| - s} \overset{\text{Lemma 6}}{\leq} \frac{d\delta^{\max}_{\text{same corner}} + s}{d - s} =: g(d)\end{aligned}$$

From the formula it is clear that $g(d)$ is monotonously decreasing in d and tends to $\delta^{\max}_{\text{same corner}}$ for $d \to \infty$. Numerically we calculate that $g(d) = \delta_{\text{critic}}(a_{\text{opt}})$ is attained at approximately 40.0158. □

It remains to prove inequality (B.2) for vertices which are close to each other. Here, we have to take into account the exact position of each vertex.

Lemma 8. *Let $p, q \in V(a_{\text{opt}})$ be vertices of the graph $G(a_{\text{opt}})$ whose Euclidean distance satisfies $|pq| \leq 50$. Then, we have $\delta_{G(a_{\text{opt}})}(p, q) < \delta_{\text{critic}}(a_{\text{opt}})$.*

Proof. By Lemma 5, we only have to consider pairs of points

$$p = p_k(0,0) \quad \text{and} \quad q = q_{k'}(i', j'), 0 \leq j' \leq i' \leq 50.$$

k	subcase	$k'=1$	$k'=2$	$k'=3$
	$\alpha = 0°$	0	$-s$	$-s$
1	$0° < \alpha < 30°$	0	$-s$	$-s$
	$\alpha = 30°$	0	0	$-s$
	$\alpha = 0°$	$+s$	0	0
2	$0° < \alpha < 30°$	$+s$	0	0
	$\alpha = 30°$	$+s$	0	0
	$\alpha = 0°$	$+s$	0	0
3	$0° < \alpha < 30°$	$+s$	0	0
	$\alpha = 30°$	$+s$	$+s$	0

Table 1. The values of the correcting term $c_\alpha(k, k')$ for all possible cases.

We will prove the lemma by simply calculating the dilation of all these pairs of points numerically and showing that it lies below $\delta_{\text{critic}}(a_{\text{opt}})$. Because of the definition of $p_k(i, j)$ in (3), by applying (2) we get

$$\begin{aligned} |pq|^2 &= |p_k(0,0)p_{k'}(i',j')|^2 \\ &= \left(i' + \frac{j'}{2} + a_{\text{opt}}\left(\cos((k'-1)120°) - \cos((k-1)120°)\right)\right)^2 \\ &\quad + \left(\frac{\sqrt{3}}{2}j' + a_{\text{opt}}\left(\sin((k'-1)120°) - \sin((k-1)120°)\right)\right)^2 \end{aligned} \tag{B.3}$$

The length of a shortest path connecting p and q in G can be bounded by

$$d_G(p,q) \leq (i'-j')(s+l) + j'(s+2l) + c_\alpha(k,k') \tag{B.4}$$

where $c_\alpha(k, k')$ is a correcting term which disappears for $k = k'$. The arguments for the formula for $k = k'$ are the same as the ones leading to the formulas for $\delta^{0°}$ and $\delta^{30°}$ in the proof of Lemma 7. The values of the correcting term are displayed in Table 1. They can easily be verified by taking a close look at the graph. As before $\alpha = 0°$ denotes the case $j' = 0$ and $\alpha = 30°$ means $j' = i'$.

The equations (B.3) and (B.4) combined with Table 1 yield a formula for $\delta(p, q) = \delta(p_k(0, 0), p_{k'}(i', j'))$ depending on k, k', i' and j'. If we consider the nine cases of $(k, k') \in \{1, 2, 3\} \times \{1, 2, 3\}$ separately, we get nine functions depending only on i' and j'. Because of Lemma 6 we do not have to consider the three cases anymore where $k = k'$. The maximum of the remaining six functions is plotted in Figure 26. We always have $\delta(p, q) \leq \delta_{\text{critic}}(a_{\text{opt}})$. Equality is only attained for $(k, k', i, j) = (1, 2, 1, 1)$ and $(k, k', i, j) = (2, 3, 1, 1)$, which both denote critical pairs. □

We summarize the results in the following theorem.

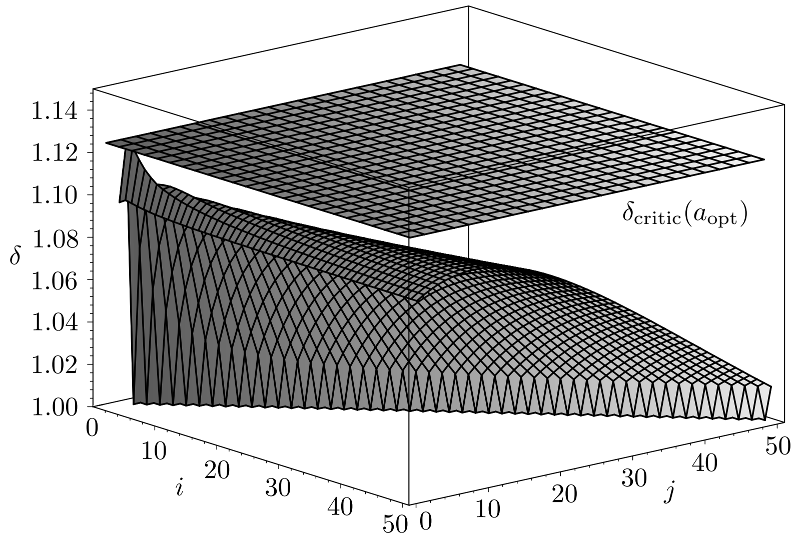

Fig. 26. The maximum of the six functions $\delta_{k,k'}(i',j') := \delta(p_k(0,0), p_{k'}(i',j'))$ for $k, k' \in \{1,2,3\}$, $k \neq k'$, lies nowhere above $\delta_{\text{critic}}(a_{\text{opt}})$.

Theorem 7. *Let $G(a)$ denote the graph constructed like described in the beginning of this section, where a is a parameter in the interval $[0, \frac{2}{3})$. Let $\delta_{\text{critic}}(a)$ denote the function defined in (B.1), and let $a_{\text{opt}} \approx 0.24857$ be the only root of the polynomial $9a^3 - 39a^2 - 15a + 6$ in the interval $[0, \frac{2}{3})$. Then, for every $a \in [0, \frac{2}{3})$ we have*

$$\delta(G(a)) \geq \delta(G(a_{\text{opt}})) = \delta_{\text{critic}}(a_{\text{opt}}) \approx 1.124667$$

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