A Generalization of Wilkie's Theorem of the Complement, and an Application to Pfaffian Closure

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Abstract

Using a modification of Wilkie's recent proof of o-minimality for Pfaffian functions, we give an invariant characterization of o-minimal expansions of $\mathbb{R}$. We apply this to construct the Pfaffian closure of an arbitrary o-minimal expansion of $\mathbb{R}$.

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0 Introduction

In this paper we improve the main result of [W96] by weakening the assumption that $\mathcal{S}$ is determined by its smooth functions.

We assume complete familiarity with [W96]. Wilkie starts with a *weak structure* $\mathcal{S}$ which is o-minimal, and passes to $\tilde{\mathcal{S}}$ ([W96], Definition 1.6), which should perhaps be called the Charbonel closure of $\mathcal{S}$. Wilkie shows that of $\mathcal{S}$ is DSF (determined by its smooth functions) then $\tilde{\mathcal{S}}$ is closed under complement, is the *structure* generated by $\mathcal{S}$, and is o-minimal.

The main action in Wilkie's paper involves his 3.6. This result involves reference to definitions of type 3.5. We observed after reading [W96] (our motivation came from our paper [KM97a] in which Sardian arguments abound) that one may modify 3.5 so that the modification of 3.6 remains true under assumptions surely weaker than his DSF.

The original 3.5 involved subsets of $\mathbb{R}^n \times \mathbb{R}_+^k$ defined by conditions on $(x_1, \dots, x_n, \epsilon_1, \dots, \epsilon_k)$ of the form

$$\exists x_{n+1} \dots \exists x_{n+k-1} \bigwedge_{i=1}^{k} f_i(x_1, \dots, x_{n+k-1}) = \epsilon_i$$

where the $f_1, \dots, f_k$ are C^∞ functions from $\mathbb{R}^{n+k-1}$ to $\mathbb{R}$ which lie in $\tilde{\mathcal{S}}$.

A close look at [W96] shows that (without use of DSF) one gets o-minimality of $\tilde{\mathcal{S}}$ provided one has, instead of 3.6, the following for each $\mathcal{A} \in \tilde{\mathcal{S}}_n$:

(3.6)*: For each $N \geq 1$ there exists $k \geq 1$, a k-modulus $\bar{\mu}$, and a set $S_N \subseteq \mathbb{R}^n \times \mathbb{R}_+^k$, which is a finite union of sets defined in the form

$$\exists x_{n+1} \dots \exists x_{n+k-1} \bigwedge_{i=1}^{k} f_i(x_1, \dots, x_{n+k-1}) = \epsilon_i$$

with the f_i C^N functions from $\mathbb{R}^{n+k-1}$ to $\mathbb{R}$ which lie in $\tilde{\mathcal{S}}$, such that $\partial\bar{\mathcal{A}} \leq \mathcal{S}(\text{mod } \bar{\mu})$ and $\mathcal{S} \leq \bar{\mathcal{A}}(\text{mod } \bar{\mu})$.

Now one looks for hypotheses on $\mathcal{S}$, weaker than DSF, which permit an inductive proof of (3.6)* along the lines of Wilkie's inductive proof of 3.6.

One such is given by:

Definition 1. A prestructure $\langle S_n : n \geq 1 \rangle$ satisfies DC^N for all N if for each $\mathcal{A} \in S_n$ there exists an $m \geq n$ such that for each N $\mathcal{A}$ is of the form

$\pi[Z(f_N)]$ where f_N is a C^N function in $\mathcal{S}$, $f_N : \mathbb{R}^m \to \mathbb{R}$, and π is the natural projection $\mathbb{R}^m \to \mathbb{R}^n$.

Obviously DSF implies DC^N for all N. Our refinement of Wilkie's result is:

Theorem 1. Suppose $\mathcal{S}$ is an o-minimal weak structure satisfying DC^N for all N. Then $\tilde{\mathcal{S}}$ is o-minimal, and is the smallest structure containing S.

We prove also a converse:

Theorem 2. Let J be an o-minimal structure. Then there is an o-minimal weak structure $\mathcal{S}$ satisfying DC^N for all N and $\tilde{\mathcal{S}} = J$.

Indeed $\mathcal{S}$ can be chosen so that it satisfies DC^N for all N in the strong sense that we can take $m = n$ in Definition 1.

Theorem 1 needs only a small modification to Wilkie's proof. Theorem 2 follows from a striking result in [DM96].

We applied Theorem 1 already in late 1996 to get o-minimality of systems got by adjoining to o-minimal $\mathcal{S}$ total C^∞ functions "Pfaffian over $\mathcal{S}$". The proof used refinements of the basic method of Khovanski [K91]. The restriction to total functions was not seen by us as essential, but the C^∞ assumption seemed hard to eliminate. This was first done by Speissegger [S97], by a totally different method.

By using the more routine part of [S97], and dispensing entirely with the "T^∞-Pfaffian" terminology, we are now able to give a simple proof of the o-minimality of Pfaffian (or, maybe better, Rolle) closure.

1 Proof of Theorem 1

Nothing in [W96] needs to be changed until 3.5, which should be replaced by $(3.5)_N$ for each N, where $(3.5)_N$ is just like 3.5 except that f_i is now assumed only to be a C^N element of $\tilde{\mathcal{S}}$.

The goal now is to show that if $\mathcal{S}$ satisfies DC^N for all N then for each $n \geq 1$, $\mathcal{A} \in \tilde{\mathcal{S}}_n$, and each $N \geq 1$:
$(3.6)_N$ There exists $k \geq 1$ (the N-complexity of $\mathcal{A}$), a k-modulus $\bar{\mu}$ (the N-modulus of $\mathcal{A}$) and a set $\mathcal{S} \subseteq \mathbb{R}^n \times \mathbb{R}^k_+$ (the N-approximation of $\mathcal{A}$) which is a finite union of sets defined by conditions of the form $(3.5)_N$ (the

N-approximating constituents of $\mathcal{A}$) such that $\partial\bar{\mathcal{A}} \leq \mathcal{S} \pmod{\bar{\mu}}$ and $\mathcal{S} \leq \bar{\mathcal{A}} \pmod{\bar{\mu}}$.

Now, Wilkie's Lemma 3.7 holds if 3.6 is replaced by $(3.6)_N$ for some fixed N.

However, the statement and proofs of the subsequent lemmas need modification, though the basic ideas remain the same.

One wants first to show for $\mathcal{A} \in \mathcal{S}_n$ that $\mathcal{A}$ satisfies $(3.6)_N$ for all N. We will use the assumption of DC^N for all N, and get the obvious analogue of 3.11. Thereafter nothing will need to be changed.

So, consider $\mathcal{A} \in \mathcal{S}_n$. Since we assume DC^N for all N, there exists $m \geq n$ such that for each N there is a C^N $g_N : \mathbb{R}^m \to \mathbb{R}$, g_N in $\mathcal{S}$, $\mathcal{A} = \pi[Z(g_N)]$.

Firstly, by inspection of the *proof* of Wilkie's Lemma 3.8, one sees that $(3.6)_N$ holds for $Z(g_N)$.

Now (the crucial step) one inspects the proof of 3.10. This shows that if $N \geq m$ then $(3.6)_{N-m}$ holds for $\mathcal{A}$ $(= \pi[Z(g_N)])$. The drop to $N - m$ comes about via the differentiations used in each application of 3.10. By replacing N by $N + m$, we conclude that $(3.6)_N$ holds for $\mathcal{A}$.

We thus have, in the obvious adaptation of Wilkie's notation,

Corollary $(3.11)_N$ Suppose $\mathcal{S}$ is o-minimal and satisfies DC^N for all N. Let $n \geq 1$, $\mathcal{A} \in \mathcal{S}_n$. Then $(3.6)_N$ holds for $\mathcal{A}$.

¿From here on, we can take up Wilkie's development without change. His 3.10 should in its general application be unwound to:
Let $n \geq 1$, $\mathcal{A} \in \tilde{\mathcal{S}}_{n+1}$, and suppose $(3.6)_{N+1}$ holds for $\mathcal{A}$. Then $(3.6)_N$ holds for $\pi[\mathcal{A}]$, where $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is the projection map onto the first n coordinates.

His Lemma 3.12 goes through with $(3.6)_N$ replacing (3.6) in its two occurences. His Theorem 3.13 becomes:

Assume $\mathcal{S}$ satisfies DC^N for all N. Let $n \geq 1$ and suppose $\mathcal{A} \in \tilde{\mathcal{S}}_n$. Then $(3.6)_N$ holds for $\mathcal{A}$, for all N.

His proof simply goes through, and his Section 4 adapts (as he essentially remarks at its outset) to our hypothesis. Theorem 1 is proved.

2 The Converse. Theorem 2

Let J be an arbitrary o-minimal *structure* on $\mathbb{R}$. The following is a remarkable result of van den Dries and Miller [DM96] (inspired by unpublished work of Bierstone, Milman and Pawlucki):

Suppose $\mathcal{A} \in J_n$, $\mathcal{A}$ closed. Then for each $j \geq 1$ there is a total C^j f in J with $\mathcal{A} = Z(f)$.

Rather more trivial is the fact that every element of J is a Boolean combination of closed sets (this follows from cell decomposition [D97]). Combining this with Theorem 3, and using the usual equivalence

$$x \neq 0 \Leftrightarrow (\exists y)(yx - 1 = 0)$$

one gets immediately

Theorem 2 Any o-minimal structure J is of the form $\tilde{\mathcal{S}}$, where $\mathcal{S}$ is an o-minimal weak structure satisfying DC^N for all N.

Proof Take $\mathcal{S} = J$.

3 The Application to Pfaffian Closure

3.1 As Wilkie remarks at the end of [W96], our method permits a relativization of his theorem on o-minimality of the structure based on the (total) classical Pfaffian functions. A first version of this was shown to Wilkie in late 1996. Later (less carefully presented) versions dealt with the case of adjoining C^∞ total functions Pfaffian over C^1 functions of an o-minimal $\mathcal{S}$. The restrictions to C^∞ and *total* are blemishes, removed by Speissegger [S97] in later, independent work relying heavily on work of Moussu - Roche [MR91] and Lion - Rolin [LR96] (the latter inspired also by [W96]). While the restriction to total was never really imposed by our original method, we faced serious difficulties in trying to remove the C^∞ assumption in certain variations on Sard's Theorem (as in Wilkie's 2.7).

In this exposition we will profit from (a small) part of Speissegger's [S97] to give a new proof of his main result, and to give a small generalization of it. We stress that without access to Speissegger's preprint [S97] we would have had to settle for a weaker result (albeit with a more perspicuous proof).

3.2 Rolle leaves (following Speissegger) Let $U \subset \mathbb{R}^n$ be open, and $\omega = a_1 dy_1 + \ldots + a_n dy_n$ a 1-form on U of class C^1. Let

$$S(\omega) = \{y \in U : a_i(y) = 0 \ 1 \leq i \leq n\}.$$

Let E be the closed subset of $(U - S(\omega)) \times \mathbb{R}^n$ defined by

$$\{(y,x) : \sum a_i(y)x_i = 0\},$$

and let $p : E \to U - S(\omega)$ be the projection to $U - S(\omega)$. Let U_j be the open set $\{y \in U : a_j(y) \neq 0\}$. Then

$$p^{-1}(U_j) = \left\{ (y,x) : y \in U_j \ , \ \hat{x}_j = -\sum_{i \neq j} \frac{a_i(y)}{a_j(y)} x_i \right\}$$

and the right hand side is homeomorphic to $U_j \times \mathbb{R}^{n-1}$ via the map φ_j given by

$$(y,x) \mapsto (y, x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n).$$

Also, if π is the projection $U_j \times \mathbb{R}^n$ to U_j, we have for $(x,y) \in p^{-1}(U_j)$, $p(y,x) = y$, and $\pi\varphi_j(y,x) = y$.

If $y \in U_j$, we have the homeomorphism

$$\varphi_{j,x} : p^{-1}(y) \ \overset{\varphi_j}{\longrightarrow} \ \{y\} \times \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$$

Now if $y \in U_i$,

$$\varphi_{i,y} \circ \varphi_{i,y}^{-1}(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) = \varphi_{i,y}(t),$$

where

$$\varphi_{i,y}(t) = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n).$$

So $\varphi_{i,x}(t) = (p(t), x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$ and $p(t) = y$.
So $t = (y, x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_n)$ some x_i
and $\varphi_{j,y}(t) = (x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n)$.

Thus $\varphi_{j,y} \circ \varphi_{i,y}^{-1}(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$ is linear and $(p, E, U \setminus S(w))$ is a vector bundle of dimension $n-1$.

An integral manifold of $w = 0$ is an $(n-1)$ dimensional immersed C^1 submanifold on which the above is the tangent bundle.

A *leaf* of $w = 0$ is a *Rolle leaf* if L is an embedded submanifold of $U \setminus S(w)$, closed in $U \setminus S(w)$, such that for each C^1 curve $\gamma : [0,1] \to U$ with $\gamma(0), \gamma(1) \in L$ there is $t \in [0,1]$ with $(a_1(\gamma(t)), \dots a_n(\gamma(t)))$, grad $\gamma(t) = 0$.

The *crucial* example is (as in Speissegger):
Example: $V \subseteq \mathbb{R}^n$, nonempty, open, connected, $f : V \to \mathbb{R}$ C^1 such that

$$\frac{\partial f}{\partial y_i}(y) = F_i(y, f(y)), \quad y \in V, 1 \leq i \leq n.$$

Then the graph $\Gamma(f)$ of f is a Rolle leaf on $U = V \times \mathbb{R}$ of

$$\omega = F_1 dy_1 + \ldots + F_n dy_n - dy_{n+1}.$$

As Speissegger remarks, there is no reason to restrict to integrable 1-forms, when one is working over an o-minimal theory (Section 2 of [S97], routine Remark).

3.3 Here is our setting. $\mathcal{S}$ is an o-minimal structure in Wilkie's sense. Let U be an open element of $\mathcal{S}_n$, and $a_1, \ldots, a_n$ C^1 functions $U \to \mathbb{R}$ in $\mathcal{S}$. Let $\omega = a_1 dx_1 + \ldots + a_n dx_n$, so clearly ω can naturally be called "in $\mathcal{S}$". Let L be a Rolle leaf for $\omega = 0$.

Let $\mathcal{S}[L]$ be the structure generated by $\mathcal{S}$ and L. Then Speissegger proved that $\mathcal{S}[L]$ is o-minimal. Iteration of these operations $\mathcal{S} \mapsto \mathcal{S}[L]$ leads to a natural notion of Pfaffian (or, perhaps better, Rolle) closure of $\mathcal{S}$.

Speissegger's proof has a distinctly clear and elementary component (Section 2), and then a longer section involving "T^∞ - Pfaffian" sets. We will show that the latter (which is hardly constructive) is unnecessary, and can be replaced by use of our Theorem 1.

3.4 Let $\mathcal{S}$ be an o-minimal structure. We pass to Rolle($\mathcal{S}$), a prestructure extending $\mathcal{S}$, where $(\text{Rolle}(\mathcal{S}))_n$ consists of all finite unions of sets

$$\mathcal{A} \cap L_1 \cap \ldots \cap L_k \qquad (*)$$

where $\mathcal{A} \in \mathcal{S}_n$, and each L_i is a Rolle leaf associated with data (U_j, ω_j) in $\mathcal{S}$.

We will show, rather easily, that Rolle($\mathcal{S}$) is an o-minimal weak structure in Wilkie's sense, and satisfies DC^N for all N.

For various reasons we need to represent sets of the form (*) as projections of sets of the same form, but where $U_j = U$ for all j. The idea is standard. We work in $(\mathbb{R}^n)^k$ and let $U = U_1 \times \ldots \times U_k$. Use coordinates $x_{ji}(1 \leq i \leq n)$ for the j^{th} copy of $\mathbb{R}^n$ in $(\mathbb{R}^n)^k$, and let

$$\tilde{\omega_j} = \sum_i a_{ji}(x_{j1}, \ldots, x_{jn}) dx_{ji}$$

where

$$\omega_j = \sum a_{ji}(x_1, \ldots, x_n) dx_i.$$

The $\tilde{\omega_j}$ are forms on U. Let

$$\tilde{L_j} = U_1 \times \ldots \times U_{j-1} \times L_j \times U_{j+1} \times \ldots \times U_n.$$

Clearly $\tilde{L_j}$ is a Rolle leaf for $\tilde{\omega_j}$ on U.

Now consider

$$\mathcal{A}^k \cap \tilde{L_1} \cap \ldots \cap \tilde{L_k} \cap \triangle,$$

where $\triangle$ is the diagonal copy of $\mathbb{R}^n$ in $(\mathbb{R}^n)^k$. This set is of form (*), and its projection to the first copy of $\mathbb{R}^n$ gives our original set.

The main point of this is that to prove uniform bounds for the number of connected components of sets (*) we can assume that the U_i are the same.

3.5 **Lemma 3** Rolle($\mathcal{S}$) is a weak structure.

Proof The product condition (WS3) follows from a variant of the argument above. (WS4), the closure under GL_n action, is obvious. □

Lemma 4 Rolle($\mathcal{S}$) satisfies (WS6).

Proof Using the final remark of 3.4, this follows from Corollary 2.7 of [S97], which has a straightforward proof. □

Lemma 5 Rolle($\mathcal{S}$) satisfies WS6.

Proof We repeatedly use

i) union commutes with projection;

ii) the class of projections (from various $\mathbb{R}^m$) of closed sets is closed under intersection.

It clearly suffices to show that any L is the projection of a closed set in Rolle($\mathcal{S}$), where L is a Rolle leaf for ω on U, where ω and U are in $\mathcal{S}_n$.

Do a cell-decomposition in $\mathcal{S}$ to express U as a finite union of open cells U_i in $\mathcal{S}$, each equipped with a C^1 homeomorphism $f_i : \mathbb{R}^n \simeq U_i$, f_i in $\mathcal{S}$.

$L_i = L \cap U_i$ is Rolle on U_i (cf. [S97], Lemma 1.4) for $\omega_i = \omega \restriction U_i$, or is $\emptyset$. Also, $\tilde{\omega_i} = (f_i)^*\omega_i$ is a C^1-form on $\mathbb{R}^n$, and $(f_i)^*L_i$ is a Rolle leaf for $\tilde{\omega_i}$.

So it suffices to prove the result for $U = \mathbb{R}^n$. Let

$$\omega = a_1 dx_1 + \ldots + a_n dx_n$$

as usual. Let $U_n = \{\bar{x} : a_n(\bar{x}) \neq 0\}$. Go to a finite decomposition (in $\mathcal{S}$) of U_n as a union of cells C^1-homeomorphic to $\mathbb{R}^n$. Pull back again, to reduce to $U = \mathbb{R}^n$, $a_n(\bar{x})$ never 0. Then it is standard that L is the graph of a C^2 function. This concludes the proof. □

Corollary Rolle($\mathcal{S}$) is o-minimal.

So $\widetilde{\text{Rolle}(\mathcal{S})}$ enjoys all the nice properties detailed by Wilkie. In particular, $\widetilde{\text{Rolle}(\mathcal{S})}$ is o-minimal, closed under partial differentiation, has the unrestricted Sard Property, etc.

Now we come to the last step which will give us the main result that Rolle($\mathcal{S}$) generates an o-minimal *structure*.

Lemma 6 Rolle($\mathcal{S}$) satisfies DC^N for all N.

Proof This is a significant refinement of the proof of the preceding lemma (which had to be done first, to exploit the o-minimality of $\widetilde{\text{Rolle}(\mathcal{S})}$).

Now we have to prove the following (exploiting the tricks detailed at the start of the previous proof):

If $U \in \mathcal{S}_n$ is open, and ω is a C^1 1-form on U, also "in" $\mathcal{S}$, and L is a Rolle leaf for ω, then there exists $m \geq n$ such that for each $N \geq 1$ there is a C^N $f_N : \mathbb{R}^m \to \mathbb{R}$ in $\tilde{\mathcal{S}}$ so that $L = \pi\,[Zer(f_N)]$, where $\pi : \mathbb{R}^m \to \mathbb{R}^n$ is the usual projection.

This time a more delicate decomposition of U is required, depending on N, and one has to ensure m remains bounded throughout.

Fix N. First decompose U (in $\mathcal{S}$) into C^N cells each C^N homeomorphic to $\mathbb{R}^n$ (in $\mathcal{S}$) [DM96], so reducing (always with attention to m) to the case U is a C^N cell. Then partition U (in $\mathcal{S}$) into a closed set D and a dense union of C^N cells on which w is a C^N form (use [DM96]). So we reduce to the cases (for U)

i) $D \cap L$;

ii) U a C^N cell, w C^N.

For the latter, pull back to $\mathbb{R}^n$, so $U = \mathbb{R}^n$. Then as in the previous proof we can reduce to $U = \mathbb{R}^n$ and L the graph of a C^N function Ψ on $\mathbb{R}^{n-1}$, so L is $\widetilde{Zer}\,(\Psi(x_1, \ldots, x_{n-1}) - x_n)$, and $\Psi(x_1, \ldots, x_{n-1}) - x_n$ is a C^N function in Rolle($\mathcal{S}$).

For the former, write D (in $\mathcal{S}$) as a disjoint union of connected manifolds $D_i \subseteq U$, of dimension $\leq n-1$. Within D_i, consider the $\mathcal{S}$-definable subset of points where the tangent space is included in ker (ω). Decompose this again into definable connected submanifolds in $\mathcal{S}$. Any one of these is either $\subseteq L$ or disjoint from L ([S97], 1.4). Thus $L \cap D$ is a finite union of sets in $\mathcal{S}$, and so is in $\mathcal{S}$. Thus it satisfies DC^N all N, and we are (essentially) done. One easily checks that the m remains bounded. □

So

Theorem 7 (Speissegger). Rolle ($\mathcal{S}$) generates an o-minimal structure.

Proof. By Theorem 1. □

3.6 (Minor) Refinements We have used the fact that the forms ω are C^1 rather systematically. However we can make some improvements, with minimal effort.

Theroem. Let $\mathcal{S}$ be o-minimal, $U \subseteq \mathbb{R}^n$ a connected open set in $\mathcal{S}$. Let $f : U \to \mathbb{R}^n$ be a C^1 function satisfying a system

$$\frac{\partial f}{\partial x_i} = P_i\left(\bar{x}, f(\bar{x})\right), \ \ 1 \leq i \leq n,$$

where the P_i are continuous, and in $\mathcal{S}$. Then f is in Rolle ($\mathcal{S}$), so in particular f lives in an o-minimal extension of $\mathcal{S}$.

Proof. Break up U (in $\mathcal{S}$) into finitely many connected open V and a D, so that the union of the V is dense in U, with complement D, and the P_i are C^1 on each V. Then $f \restriction V$ is in Rolle ($\mathcal{S}$), and the graph of f on U is got by closure.

In particular,

Corollary (Speissegger) Rolle ($\mathcal{S}$) is closed under integration of continuous functions of one variable.

4 Concluding Remarks

We came to this topic from our very explicit work [KM97a] on bounds for Vapnik-Chervonenkis dimension of semi-Pfaffian families. There one made constant appeal to Sardian arguments. The power of the idea there convinced us that a "Sardian" approach to o-minimality would be fruitful. The work of Charbnel and Wilkie certainly confirms this.

Our 1996 work gave results significantly weaker than those reported here, though we could do the closure under integration of the last corollary. Speissegger got the optimal result, using the quite heavily disguised version of Wilkie's technology due to [LR96]. Our proof here is explicitly in the style of Wilkie.

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