Randomized Complexity of Linear Arrangements and Polyhedra

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Abstract

We survey some of the recent results on the complexity of recognizing n–dimensional linear arrangements and convex polyhedra by randomized algebraic decision trees. We give also a number of concrete applications of these results. In particular, we derive first nontrivial, in fact quadratic, randomized lower bounds on the problems like Knapsack and Bounded Integer Programming. We formulate further several open problems and possible directions for future research.

1 Introduction.

Linear search algorithms, algebraic decision trees, and computation trees were introduced early to simulate random access machines (RAM) model. They are also a very useful and simplified abstraction of various other RAM–related computations cf. [AHU74], [DL78], [Y81], [SP82], [M84], [M85a], [KM90], and a useful tool in computational geometry. The same applies for the randomized models of computation. Starting with the papers of Manber and Tompa [MT85], Snir [S85], Meyer auf der Heide [M85a], [M85c] there was an increasing interest, and continuing effort in the last decade to understand the intrinsic power of randomization in performing various computational

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tasks. We refer to Bürgisser, Karpinski and Lickteig [BKL93], Grigoriev and Karpinski [GK94], Grigoriev, Karpinski, Meyer auf der Heide and Smolensky [GKMS97], and Grigoriev, Karpinski and Smolensky [GKS97] for the recent results (for the corresponding situation in a randomized bit model computation cf., e.g., [KV88], [FK95]). For some new randomized lower bounds on high dimensional geometric problems see also Borodin, Ostrovsky and Rabani [BOR99]. In the retrospective, several algebraic and topological methods introduced for proving lower bounds for deterministic algebraic decision trees turned out to fail utterly for some reason for the randomized model of computation, see the papers on deterministic methods by Yao [Y81], [Y92], [Y94], Steele and Yao [SY82], Ben-Or [B-O83], Björner, Lovász and Yao [BLY92] and Grigoriev, Karpinski and Vorobjov [GKV97]. With the exception of some early results of Bürgisser, Karpinski and Lickteig [BKL93], and Grigoriev and Karpinski [GK93] there were basically no methods available for proving lower bounds on the depth of general randomized algebraic decision trees. In Meyer auf der Heide [M85a], a lower bound has been stated on the depth of randomized linear decision trees (with linear polynomials only) recognizing a linear arrangement. A gap in the proof of the Main Lemma of this paper was closed for the generic case first by Grigoriev and Karpinski in [GK94].

In this paper we survey some of the new methods which yield for the first time nontrivial lower bounds on the depth of randomized algebraic trees. The paper is organized as follows. In Section 2 we give necessary preliminaries for a general reader, and in Section 3 we introduce the underlying models of computation. In Section 4 we formulate the Main Results, and give some concrete applications. Section 5 deals with the phenomenon of a randomized speedup, and an explicit separation of deterministic and randomized depth. Section 6 presents some extensions of the results of Section 4. In Section 7 we formulate some open problems and possible directions for future research.

2 Preliminaries.

We refer a general reader to [G67] for basic notions on convex polytopes and linear arrangements, and to [L84] for basic algebraic notions. We refer also to [M64] for basic facts on real varieties and Betti numbers.

For $x, y \in \mathbb{R}^n$ we denote by $<x, y>$ the *scalar product* of x and y,

$$<x, y>= \sum_{i=1}^{n} x_i y_i.$$

A *hyperplane* $H \subseteq \mathbb{R}^n$ is a set defined by $H = \{x \in \mathbb{R}^n \,|\, <x, y>= \alpha\}$ for some $y \in \mathbb{R}^n$, $y \neq 0$, and $\alpha \in \mathbb{R}$. A *closed halfspace* $H \subseteq \mathbb{R}^n$ is defined by $H = \{x \in \mathbb{R}^n \,|\, <x, y> \geq \alpha\}$ for some $y \in \mathbb{R}^n$, $y \neq 0$, and $\alpha \in \mathbb{R}$.

We call a *finite union* $S = \bigcup_{i=1}^{m} H_i$ of *hyperplanes* H_i, a *linear arrangement*, and a *finite intersection* $S^+ = \bigcap_{i=1}^{m} H_i^+$ of *closed halfspaces* H_i^+, a *polyhedron*. A *k-face* L of a linear arrangement S is a *k-dimensional plane* defined by intersecting $n-k$ of the hyperplanes H_i. If L is k-dimensional on the boundary of S^+, is is also a *k-face* of S^+. We call a 0–face, a *vertex*.

When $S \subseteq \mathbb{R}^n$ is considered here as a *topological space*, it is with a subspace topology induced by $\mathbb{R}^n$. For any topological space S and an integer $k \geq 0$, $\beta_k(S)$ denotes the *i-th Betti number*, i. e., the *rank* of the *i-th singular homology* group. The *Euler characteristic* $\chi(S)$ of S is defined by $\chi(S) = \sum_k (-1)^k \beta_k(S)$ provided the Betti numbers of S are finite.

Milnor [M64] and Thom [T65] give fundamental bounds on the sums of Betti numbers $\sum_k \beta_k(S)$ of algebraic sets in $\mathbb{R}^n$ in the function of a degree bound on their defining polynomials: $\sum_k \beta_k(S) \leq d(2d-1)^{n-1}$.

We consider in sequel the following n-dimensional restrictions of NP-complete problems (cf. [DL78], [M84], [M85b]).

A *Bounded Integer Programming Problem* is a problem of *recognizing* a set

$$L_{n,k} = \{x \in \mathbb{R}^n \mid \exists\, a \in \{0, \ldots, k\}^n \,[< x, a >= k]\}$$

for a given bound k on the size of *integer solutions*.

The well known *Knapsack Problem* is the problem of recognizing the set $L_{n,1}$.

We consider further the problems of *Element Distinctness*, *Set Disjointness* and the *Resultant (Decision Version)* (cf. [B-O83]).

The *Element Distinctness* problem is the problem of recognizing the complement of the set

$$\{x \in \mathbb{R}^n \mid \exists\, i, j, i \neq j\, [x_i = x_j]\}\,.$$

The *Set Disjointness* problem is the problem of determining for given two sets $A = \{x_1, \ldots, x_n\}$, $B = \{y_1, \ldots, y_n\} \subseteq \mathbb{R}$ whether or not $A \cap B = \emptyset$, i.e. recognizing the set $\{(x_1, \ldots, x_n,\, y_1, \ldots, y_n) \in \mathbb{R}^{2n} \mid \forall\, i, j\, [x_i \neq x_j]\}$

The *Resultant* problem is the problem of computing for given $x, y \in \mathbb{R}^n$ the *resultant* of x and y, $\prod_{i,j}(x_i - y_j)$ (cf. [B-O83]). Any algorithm for the *Resultant* problem can check whether the resultant $\neq 0$, i.e. whether the sets $\{x_i\}$ and $\{y_i\}$ are disjoint, and therefore solve the *Set Disjointness* problem as well.

It is not difficult to prove that the number of *vertices (0-faces)* of the Bounded Integer Programming Problem $L_{n,k}$ is at least $(k+1)^{\frac{n^2}{16}}$, and the number of $\frac{n}{2} - faces$ (assuming n is even) of the *Element Distinctness* is $(\frac{n}{2})!$ (cf. [GKMS97]).

3 Computational Models.

We introduce now our underlying model of *randomized computations, a randomized algebraic decision tree* (RDT).

An *algebraic decision tree* of degree d (d–DT) over $\mathbb{R}^n$ is a *rooted ternary* tree. Its root and inner nodes are labelled by real multivariate polynomials $g \in \mathbb{R}[x_1, \ldots, x_n]$ of degree at most d, its leaves are marked *"accepting"* or *"rejecting"*. A *computation* of a d–DT on an input $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ consists of a sequence of traverses of a tree from the root to a leaf, always choosing the *left/middle/right* branch from a node labelled by a polynomial g according to the *sign* of $g(x_1, \ldots, x_n)$ (*smaller/equal/greater than* 0). The *inputs* $x \in \mathbb{R}^n$ arriving at accepting leaves *form* the set $S \subseteq \mathbb{R}^n$ *recognized* (or *computed*) by the d–DT.

In this paper we deal with *randomized algebraic decision trees* of *degree* d (d*–RDTs*). A d*–RDT* over $\mathbb{R}^n$ is a finite collection $T = \{T_\alpha\}$ of d*–DTs* T_α with the assigned rational probabilities p_α, $\sum_\alpha p_\alpha = 1$, of *choosing (or randomized compiling)* T_α out of the set $\{T_\alpha\}$.

It is easily seen that the above model is equivalent to the other variant of a randomized algebraic decision tree allowing coin tosses at the special *random* nodes, and not charging for the random bits used. Our model of a d*–RDT* is also easily to be seen equivalent (up to a constant factor in depth) to the "equal probability" model with all trees T_α having equal probabilities $p_\alpha = \frac{1}{|\{T_\alpha\}|}$. The rest of the paper uses this simplified "equal probability" model, and identifies a d*–RDT* with a finite collection $\{T_i\}$ of d*–DTs*. We say that such a d*–RDT recognizes (or computes)* a set $S \subseteq \mathbb{R}^n$, if it classifies every $x \in \mathbb{R}^n$ correctly (with respect to S) with probability at least $1 - \varepsilon$ for some $0 < \varepsilon < \frac{1}{2}$. The parameter ε bounds an error probability of *computations* of a d*–RDT*.

It is readily seen that the class of sets $S \subseteq \mathbb{R}^n$ *recognizable* by d*–RDTs* is *closed* under the complement.

The depth of $T = \{T_i\}$ is the *maximum* depth of T_i's in T. It is straightforward to verify that the class of sets $S \subseteq \mathbb{R}^n$ recognizable by d*–RDTs* is depth-invariant under changes of the error probability ε in the interval $(0, \frac{1}{2})$: for any two $\varepsilon_1, \varepsilon_2 \in (0, \frac{1}{2})$, if $S \subseteq \mathbb{R}^n$ is *recognized* by a d*–RDT* with error probability ε_1, and depth t, it is also recognizable by a d*–RDT* with error probability ε_2 and depth $O(t)$ [M85c]. It is also known that a d*–RDT* with a *worst case expected depth* t, a notion used by some authors, can be simulated by a d*–RDT* with depth $O(t)$ ([MT85]).

4 Main Results.

We shall deal here with the randomized complexity of linear arrangements, and convex polyhedra. For the first class of sets several topological methods were developed for obtaining lower bounds for deterministic algebraic decision trees, and deterministic computation trees cf. [DL78], [SY82], [B-O83], [BLY92], [Y92] and [Y94]. In Ben-Or [B-O83] a general deterministic lower bound $\Omega(\log C)$ was proven for C being the number of connected components of $S \subseteq \mathbb{R}^n$ or its complement. Yao [Y92] (see also Björner, Lovász and Yao [BLY92]) proved a decade later a deterministic lower bound $\Omega(\log \chi)$ for χ the Euler characteristic of $S \subseteq \mathbb{R}^n$. A stronger lower bound $\Omega(\log B)$ for B the sum of the Betti number of $S \subseteq \mathbb{R}^n$ was proven later in Yao [Y94]. We have obvious inequalities $C, \chi \leq B$. For the second class of sets, convex polyhedra, the above topological methods fail because the invariant $B = 1$. For this class of sets, Grigoriev, Karpinski and Vorobjov [GKV97] introduced a drastically different method of counting the number of faces of $S \subseteq \mathbb{R}^n$ of all dimensions. The new method transforms a set $S \subseteq \mathbb{R}^n$ via "infinitesimal perturbations" into a smooth hypersurface and uses certain new calculus of principal curvatures on it. The resulting lower bound was $\Omega(\log N)$ for N being the number of faces of all dimensions provided N was *large enough.*

All the above mentioned methods did not work, and this for the both classes of sets on the randomized algebraic decision trees, and this for a fundamental reason. In fact, they even did not seem to work for the linear decision trees; the gap in the proof of Meyer auf der Heide [M85a] was firstly closed for the generic case by Grigoriev and Karpinski in [GK94]. The first very special randomized lower bounds were proven in Bürgisser, Karpinski and Lickteig [BKL93], and Grigoriev and Karpinski [GK93].

In this paper we survey some new general methods for proving lower bounds for *d–RDTs* recognizing linear arrangement and convex polyhedra.

Let $H_i \subseteq \mathbb{R}^n$, $1 \leq i \leq m$, $n \leq m$ be the hyperplanes, and $H_i^+ \subseteq \mathbb{R}^n$, $1 \leq i \leq m$, $n \leq m$, the closed halfspaces. Define $S = \bigcup_{i=1}^m H_i$, a linear arrangement, and $S^+ = \bigcap_{i=1}^m H_i^+$, a polyhedron.

In [GKMS97] the following general theorem was proven.

Theorem 1. ([GKMS97]). *Let $\varepsilon, c, \zeta, \delta$ be any constants such that $0 \leq \varepsilon < \frac{1}{2}$, $c > 0$, and $\zeta > \delta \geq 0$. There exists a constant $c^* > 0$ with the following property. If S (S^+) has at least $m^{\zeta(n-k)}$ k–faces for* certain *$0 \leq k < n$, then the* depth *of any d-RDT computing S (S^+) with the error probability ε is greater than $c^*(n-k)\log m$ for any degree $d < cm^\delta$.*

The original idea of this paper uses a nonarchimedean extension of a field, and consequently Tarski's transfer principle [T51], and a *leading term sign* technique combined with a *global labelled flag* construction (attached to all *k-faces* along the path of a decision tree) for counting number of faces of all dimensions of the set S.

We recall now the bounds of Section 2 on the number of *k-faces* of the n-dimensional restrictions of NP-complete problems. The result above yields directly the following concrete applications towards a *Bounded Integer Programming Problem*

$$L_{n,k} = \{x \in \mathbb{R}^n \mid \exists a \in \{0, \ldots, k\}^n [< x, a >= k]\} \text{ (cf. [M85a])},$$

and the *Knapsack Problem* $K_n = L_{n,1}$ (cf. [M85b].

Corollary 1.

(i) $\Omega(n^2 \log(k+1))$ *is a lower bound for the depth of any d-RDT computing the Bounded Integer Programming Problem* $L_{n,k}$.

(ii) $\Omega(n^2)$ *is a lower bound for the depth of any d-RDT computing the Knapsack Problem.*

Theorem 1 gives in fact much stronger lower bounds for non-constant degree d, d*–RDTs*. In the first case the sufficient condition on degree is $d = \Omega((k+1)^{\delta n})$ for $\delta < \frac{1}{16}$, in the second case $d = \Omega(2^{\delta n})$ for $\delta < \frac{1}{16}$.

It is also not too difficult to derive further randomized lower bounds (cf. [GKMS97]).

Corollary 2. $\Omega(n \log n)$ *is a lower bound for the depth of any d-RDT computing any of the following problems:*

(i) *Element Distinctness,*

(ii) *Set Disjointness,*

(iii) *Resultant.*

Corollary 2 holds also for the non-constant degree d–$RDTs$ with $d = \Omega(n^\delta)$ for $\delta < \frac{1}{2}$ (cf. [GKMS97]). This leads us again to the very interesting computational issue of the dependence of the actual computational power of d–$RDTs$ on the degree bound d.

It is also interesting to note that the proof method of [GKMS97] gives a new elementary technique for deterministic algebraic decision trees without making use of Milnor-Thom bound on Betti numbers of algebraic varieties.

5 Randomized Speedup.

We shall investigate now the computational power of linear degree and sub-linear depth n–$RDTs$ and compare it with deterministic n–DTs. Such models can be easily simulated by *randomized algebraic computational trees* (CTs) in linear time. Also, it is easy to see that linear time CTs and linear time randomized CTs correspond to the non-uniform deterministic linear time and randomized linear time classes on the real number machine models (cf. [CKKLW95]).

Let us consider now the following permutational problem $PERM(a) = \{x | x \in \mathbb{R}^n,\ x \text{ is a permutation of } a\}$ for $a = (a_1, ..., a_n) \in \mathbb{R}^n$, $a_i \neq a_j$ for $i \neq j$. The number of connected components of $PERM(a)$ equals $n!$ and of its complement equals 1. By Ben-Or [B-O83] the lower bound of any deterministic CT or any $n - DT$ computing $PERM(a)$ is $\Omega(n \log n)$. However as noticed in [BKL93], there exists an n–RDT of constant depth computing $PERM(a)$ as follows. Construct a polynomial $p(\zeta) = \prod_{i=1}^n (\zeta - a_i) - \prod_{i=1}^n (\zeta - x_i) \in \mathbb{R}[\zeta]$. We have $x = (x_1, ..., x_{n)} \in PERM(a)$ iff $p(\zeta) \equiv 0$. The identity $p(\zeta)$ can be checked probabilistically by randomly chosing ζ from the set $\{1, ..., 4n\}$ and verifying whether $p(\zeta) = 0$. If $p(\zeta) = 0$, we decide that $p(\zeta) \equiv 0$ and $x \in PERM(a)$, otherwise we have a witness that $P(\zeta) \not\equiv 0$ and $x \notin PERM(a)$. The error probability is bounded by $\frac{1}{4}$. Construct now $4n$ many n–DTs T_ζ having a single decision element $p(\zeta)$, $T = \{T_\zeta\}_{\zeta \in \{1,...,4n\}}$. T computes $PERM(a)$ with error probability $\frac{1}{4}$.

Lemma 1. ([BKL93]). *There are problems $S \subseteq \mathbb{R}^n$ computable in $O(1)$ depth on n-RDTs which are not computable by any n-DT in depth $o(n \log n)$*

The next separation results will be much more powerful in nature. We extend our underlying decision tree models to allow arbitrary analytic functions as decision elements (cf. [R72]). We denote such decision trees by *A–DTs*, and *A–RDTs*, respectively.

Let us consider now the Octant Problem $\mathbb{R}^n_+ = \{(x_1, ..., x_n) \in \mathbb{R}^n | x_1 \geq 0, ..., x_n \geq 0\}$, the problem of *testing* the membership to $\mathbb{R}^n_+$.

Rabin [R72] proved the following (see also [GKMS97])

Lemma 2. ([R72]). *Any A-DT computing $\mathbb{R}^n_+$ has depth at least n.*

Grigoriev, Karpinski, Meyer auf der Heide and Smolensky [GKMS97] were able to prove the following degree hierarchy result on randomized decision trees.

Lemma 3. ([GKMS97]). *The depth of any d-RDT computing $\mathbb{R}^n_+$ with error probability $\epsilon \in (0, \frac{1}{2})$ is greater than or equal to $\frac{1}{d}(1-2\epsilon)^2 n$.*

The Octant Problem is closely related to the well known MAX Problem: given n real numbers $x_1, \ldots, x_n$, $x_i \in \mathbb{R}$, compute the maximum of them. Rabin [R72] proved a sharp bound $n-1$ on depth of any A–DT computing MAX. Ting and Yao [TY94] proved a dramatic improvement on the depth of the *randomized* algebraic decision trees computing MAX for the case of pairwise distinct numbers (the leaves of a decision tree are labelled now by numbers $1, \ldots, n$).

Theorem 2. ([TY94]). *There exists an n-RDT computing MAX problem for the case of pairwise distinct numbers in depth $O(log^2 n)$.*

We notice that the problem on whether $x_1 = max\{x_1, ..., x_n\}$ is *equivalent* to the test whether $(x_1 - x_2, ..., x_1 - x_n)$ belongs to the octant $\mathbb{R}^{n-1}_+$.

Grigoriev, Karpinski and Smolensky [GKS97] were able to extend the assertions of Lemma 3 and Theorem 2 to the following.

Theorem 3. **([GKS97]).** *There exists an n-RDT computing* $\mathbb{R}_{+}^{n}$ *or deciding whether* $x_i = \max\{x_1, \ldots, x_n\}$ *in depth* $O(\log^2 n)$.

Theorem 4. **([GKS97]).** *There exists an n-RDT computing MAX in depth* $O(\log^5 n)$.

One notices a remarkable exponential *randomized speed–up* for the above problems having all (!) deterministic linear lower bounds, and this even for the general analytic decision trees ([R72]). An important issue remains whether the randomized speed–up can be carried even further. Interestingly, Wigderson and Yao [WY98] proved the following result connected to the construction of [TY94].

Assume that the decision tree performs only tests of the form "$x < V$", x is *smaller* than all elements in V. We call it a *subset minimum test.* The test of this form was used in the design of [TY94]. We denote a corresponding randomized decision tree (using the subset minimum test only) by *SM–RDT*.

Theorem 5. **([WY98]).** *Every SM-RDT computing MAX problem has depth* $\Omega(\log^2 n / \log\log n)$.

We turn now to the problems of proving lower bounds on the size of algebraic decision trees. Theorem 4 entails the subexponential size of *n–RDTs* computing MAX.

In this context Grigoriev, Karpinski and Yao [GKY98] proved the first exponential deterministic size lower bound on (ternary) algebraic decision trees for MAX. It should be noted that there was no size lower bound greater than $n - 1$ known before.

The method used in this paper depends on the analysis of the so called "touching frequency" of the sets computed along the branches of a decision tree with the special "wall sets" related to the cellular decomposition of the set of $(x_1, \ldots, x_n) \in \mathbb{R}^n$ satisfying $x_1 = max\{x_1, \ldots, x_n\}$.

Theorem 6. **([GKY98]).** *Any (ternary) algebraic decision tree of degree d computing MAX problem in dimension n has size* $\Omega(2^{c(d)n})$ *for the constant* $c(d) > 0$ *depending only on d.*

Grigoriev, Karpinski and Yao [GKY98] discovered also a new connection between a cellular decomposition of a set $S \subseteq \mathbb{R}^n$ defined by polynomial constraints of degree d and the maximum number of minimal cutsets $m_{d,n}$ of any rank–d hypergraph on n vertices.

Theorem 7. ([GKY98]). *Any (ternary) algebraic decision tree of degree d computing MAX problem in dimension n has size at least $2^{n-1}/m_{d,n-1}$.*

Interestingly, Theorem 7 gives improvements of the constants $c(d)$ used in Theorem 6. For any *2-DT* computing MAX problem, $c(d)$ computed via Theorem 7 is ≈ 0.47, and via Theorem 6 is ≈ 0.18 (cf. [GKY98]).

We are still lacking basic general methods for proving nontrivial lower bounds on the size (number of inner nodes) of both *d–DTs*, and *d–RDTs* with an exception of linear decision trees. In most cases the topological, and face counting methods cannot even deal with the questions about the size lower bounds of the very weak form: "is the size $t+1$ necessary?" for t a known lower bound on the depth of algebraic decision trees.

6 Extensions.

We will turn now to the model of a randomized computation tree (RCT) modeling straight line computation in which we charge for each arithmetic operation needed to compute its decision elements (cf. [B-O83]).

The papers [GK97], [G98] generalize the results of Section 4 to the case of $RCTs$ using some new results on the *border* (generalization of the *border rank* of a tensor) and *multiplicative* complexity of a polynomial.

Theorem 8. ([GK97], [G98]).

(i) $\Omega(n^2 \log(k+1))$ *is a lower bound for the depth of any RCT computing the bounded Integer Programming Problem* $L_{n,k}$.

(ii) $\Omega(n^2)$ *is a lower bound for the depth of any RCT computing the Knapsack.*

(iii) $\Omega(n \log n)$ *is a lower bound for the depth of any RCT computing the Element Distinctness.*

An important issue remains, and this in both cases, deterministic and randomized, about the generalization of algebraic decision trees and computation trees to the "ultimate models" of branching programs obtained by merging together equivalent nodes in a decision tree. An extended research on the boolean model of a branching program was carried throughout the last decade (cf., e.g., Borodin [B93], Razborov [R91] for deterministic programs, and Karpinski [K98a], [K98b], Thathachar [T98] for randomized ones). Much less is known about the model of algebraic branching programs, see also Yao [Y82].

7 Open Problems and Further Research.

An important issue of the tradeoffs between the size and the depth of algebraic decision trees, computational trees, and branching programs remains widely open. We are not able at the moment, as mentioned before, to prove any nontrivial lower bound on the size of algebraic decision trees for the n-dimensional restrictions of NP-complete problems like *Knapsack* or *Bounded Integer Programming* (cf. [M84], [M85b], [M93]). Nor can we prove any randomized size upper bounds for these problems better than the best known deterministic ones. For the recent randomized lower bounds for the *Nearest Neighbor Search* Problem on the related cell probe model see also [BOR99]. It will be very interesting to shed some more light on this model and also other related models capturing hashing and reflecting storage resources required by an actual geometric computation.

Major problems remain open about the randomized decision complexity of concrete geometric problems expressed by *simultaneous positivity* of small degree polynomials, like quadratic or cubic ones, or the *existential* problems of simultaneous positivity of small degree polynomials, corresponding to an algebraic version of the SAT problem.

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