APPROXIMATING BOUNDED METRIC TSP AND RELATED PROBLEMS.

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TRAVELING SALESMAN PROBLEM (TSP):

GIVEN A COLLECTION OF CITIES AND A DISTANCE FNCT. $d(\cdot,\cdot)$, FIND SHORTEST TOUR VISITING EACH CITY EXACTLY ONCE.

ONE OF THE OLDEST OPTIMIZATION PROBLEMS ($\rightarrow$ COMING BACK TO ~1832).

- NONAPPROXIMABLE FOR ARBITRARY DISTANCE FNCT'S.
- APPROXIMABLE FOR METRIC DIST. FNCT'S WITHIN A.R. 1.5

HARDNESS RESULTS.

- EXACT SETTING:

  NP-HARD [KARP '72]

  EVEN FOR

  (1,2)-METRIC INST.

- APPROX. SETTING:

  NP-HARD TO APPROX

  TO WITHIN TO

  $\boxed{1.0045}$.

  [PAPADIMITRIOU, VEMPALA '02]

SPECIAL INTEREST IN BOUNDED METRIC INSTANCES OF TSP:

(1,B)-METRIC HAS DISTANCES BETWEEN 1 AND B (ALL INTEGERS).

NOTATION: (1,B)-TSP

KNOWN APPROX. HARDNESS RESULTS FOR (1,B)-TSP:

- (1,8)-TSP HARD TO APPROX TO WITHIN $\boxed{1.0025}$
- (1,2)-TSP HARD TO APPROX TO WITHIN $\boxed{1.0013}$

[ENGEBRETSEN, KARPINSKI '04]

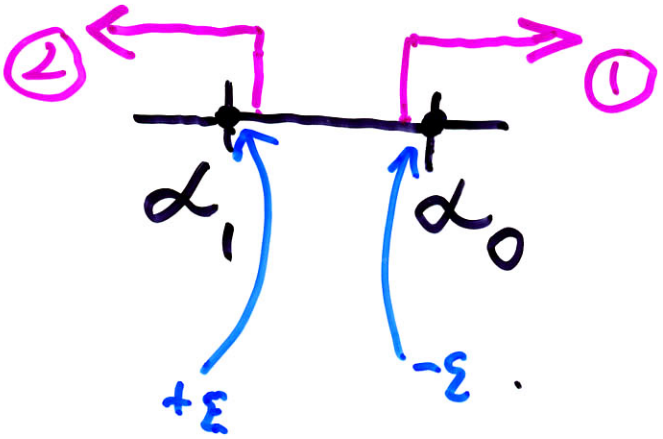
PROOFS BASED ON APPROX. HARDNESS

RESULT ON THE HYBRID EQUATIONS PROBLEM (LINEAR EQUATIONS MOD 2 WITH 2 OR 3 VAR'S PER EQUATION; EACH VAR. OCCURS 3 TIMES): HARDNESS

$$A.R. = \underbrace{\frac{62}{61} - \varepsilon}$$.

EXPLICIT PCP-METHOD:

$\Rightarrow$ HGL-METHOD:



$\Rightarrow$ A.R. $= \underbrace{\dfrac{\alpha_0}{\alpha_1} - \varepsilon}$ $(\forall \varepsilon > 0)$

$\boxed{\text{① OR ② NP-HARD TO DECIDE.}}$

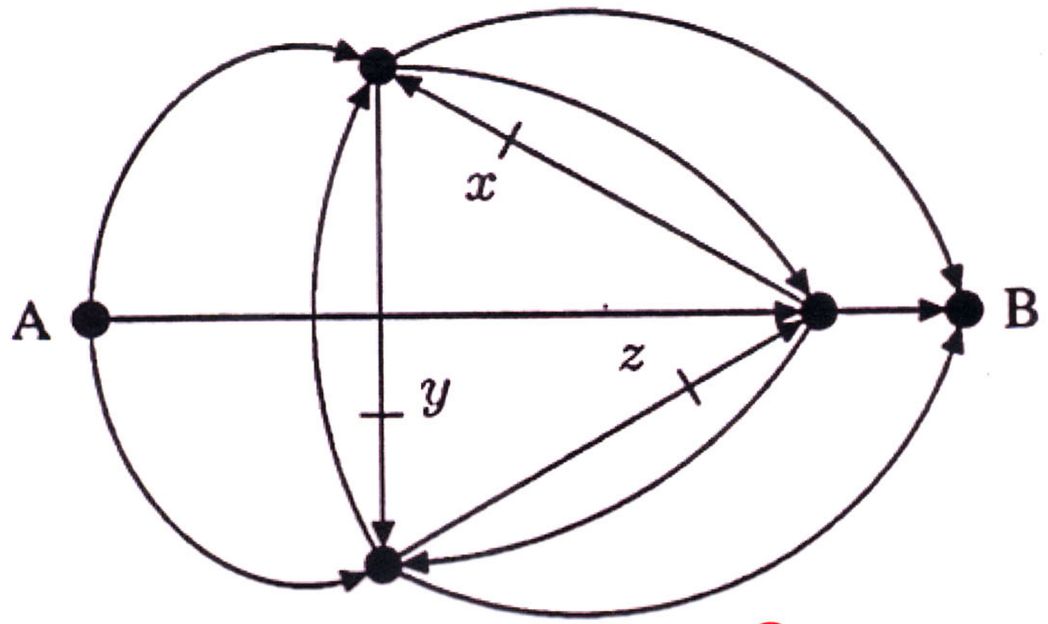
MAPPING INSTANCES OF HYBRID EQUATIONS INTO INSTANCES OF $\boxed{(1,8)\text{-TSP}}$ AND $\boxed{(1,2)\text{-TSP}}$

– TRANSLATING EACH CONSTRAINT (EQ.) LOCALLY BY GADGET.

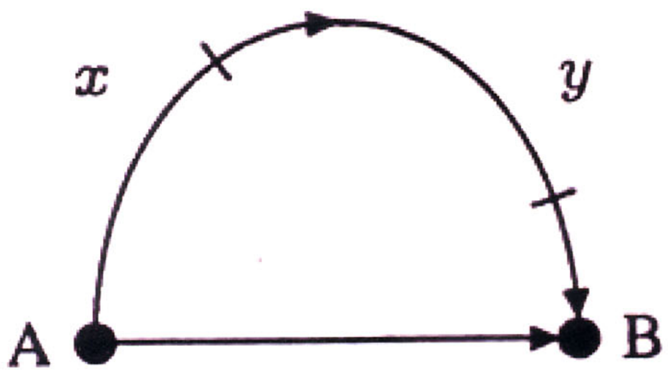
LOCAL TRANSLATIONS.

$$\boxed{x \oplus y \oplus z = 0}$$

INTO



HAMILTONIAN PATHS
A → B TRAVERSE ONLY
EVEN NUMBER OF TICKED EDG.'s



FOR $\boxed{x \oplus y = 0}$

(1,2)-TSP

IMPROVED APPROX. ALGORITHM.

(THE BEST UP TO NOW ALG. - WAS 7/6 APPROX. ALG. OF PAPADIMITRIOU AND YANNAKAKIS '93)

USING HARDVIGSEN'S ALG. FOR 2-MATCHINGS.

6)

?

GOOD UPPER APPROX. BOUNDS FOR (1,B)-TSP FOR SMALL B'S, ESPECIALLY FOR B=2 AS THIS CONNECTS DIRECTLY TO THE PATH COVER PROBLEMS, AND VARIOUS APPLICAT. IN CODE OPTIMIZ.; ...

NEW ALG.

WE REPRESENT AN INSTANCE OF (1,2)-TSP AS A GRAPH

G=({POINTS OF METRIC},
{PAIRS OF POINTS OF DISTANCE 1})

LET C BE A PATH COVER OF G WITH K PATHS (SIMPLE & NODE-DISJOINT)

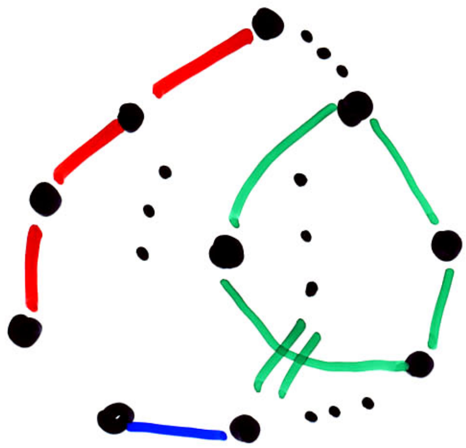
NEW ALG.

WE REPRESENT AN INSTANCE OF (1,2)-TSP AS A GRAPH

$G = (\{\overbrace{\text{POINTS OF METRIC}}^{n}\},$

$\{\text{PAIRS OF POINTS OF DISTANCE } 1\})$

LET C BE A PATH COVER OF G WITH k PATHS (SIMPLE & NODE-DISJOINT)

C:  K PATHS

- PATHS IN C HAVE $\underbrace{n-K}$ EDGES.
- WE CAN CONNECT THEM USING EDGES OF LENGTH 2 INTO A TOUR OF THE INST. OF COST $\underbrace{n+K}$.
- SO OUR PROBLEM IS TO MINIMIZE $\underbrace{n+K}$.

SUPPOSE AN OPTIMAL SOLUTION HAS COST

$$\boxed{n+k^*}$$

AND OUR GOAL IS TO APPROXIMATE WITHIN A FACTOR $8/7$.

FOR THAT IT IS SUFFICIENT TO FIND A PATH COVER WITH NO MORE THAN

$$\boxed{\frac{1}{7}n + \frac{8}{7}k^*}$$ PATHS.

AN ALGORITHM:

BASED ON AN IDEA OF "SMALL_STEP_IMPROVEMENTS" OF TENTATIVE SOLUTIONS (2-MATCHINGS) A.

NOTATION:

- k_A = # OF PATHS & CYCLES IN A;
- m_A = # OF NODES IN CYCLES;
- s_A = # OF SINGLETONS IN A.

SSI BASIC STEP:

COMPUTE $A \oplus C$, FOR A NEW EDGE SET C SUCH THAT:

1. $A \oplus C$ IS A 2-MATCHING;
2. EITHER ($\bullet\ K_{A \oplus C} < K_A$)
   OR ($\bullet\ K_{A \oplus C} = K_A$ AND $m_{A \oplus C} > m_A$)
   OR ($\bullet\ K_{A \oplus C} = K_A$ AND $m_{A \oplus C} = m_A$ AND $S_{A \oplus C} < S_A$).

3

WE SAY THAT C IMPROVES A IF THE CONDITIONS 1. AND 2. ARE SATISFIED.

SUPPOSE THERE IS A CONSTANT B SUCH THAT THE FOLLOWING IS TRUE:

(*) EITHER $\underbrace{k_A \leq \frac{1}{7}n + \frac{8}{7}k^*}$ OR THERE EXISTS C THAT IMPROVES A AND $|C| \leq B$.

THE **ALGORITHM**

B-**IMPROVE**:

$A := \emptyset$,

WHILE YOU **CAN** FIND

C OF SIZE AT

MOST B THAT

IMPROVES A -

REPLACE A

WITH $\boxed{A \oplus C}$,

OUTPUT A.

\# OF IMPROVEMENTS $< n^3$.

EACH SEARCH FOR IMPROVEMENTS TAKES POLY TIME DEPENDING ON B.

THEOREM. (*) IS TRUE FOR B=15.

RUNNING TIME ESTIMATE : $\underbrace{O(n^9)}$

$\underbrace{O(n^3)}$ - FOR SPARSE INSTANCES

$\Downarrow$

- THERE EXISTS A POLY TIME APPROX. ALG. FOR (1,2)-TSP WITH APPROX. RATIO $\underbrace{8/7}$.

A SLIGHTLY MODIFIED ALG. GIVES ALSO POLY TIME APPROX. ALGs FOR THE MAX PATH COVER (GIVEN A GRAPH, CONSTRUCT A SET OF NODE DISJOINT PATHS MAXIMIZING THE # OF EDGES) AND MAX-{0,1} TSP WITH A.R. $\underbrace{7/6}$.

17'

METHOD:

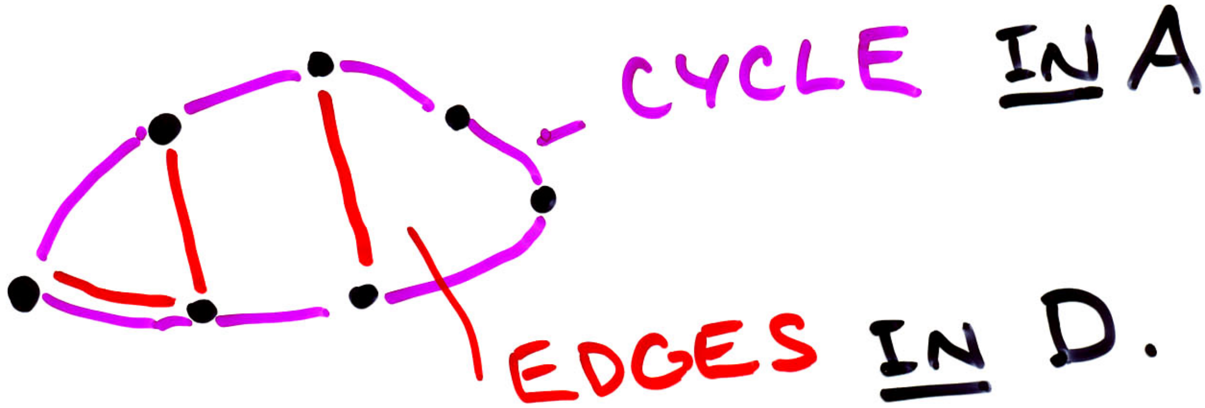
CONSECUTIVE PATH COVER IMPROVEMENTS, AND COLOR ALTERNATING PATH DECOMPOSITIONS.

PROOF IDEA OF (*)-THEOREM.

LET T BE AN OPTIMUM SOLUTION $K_T = k^*$, AND A BE A 2-MATCHING.

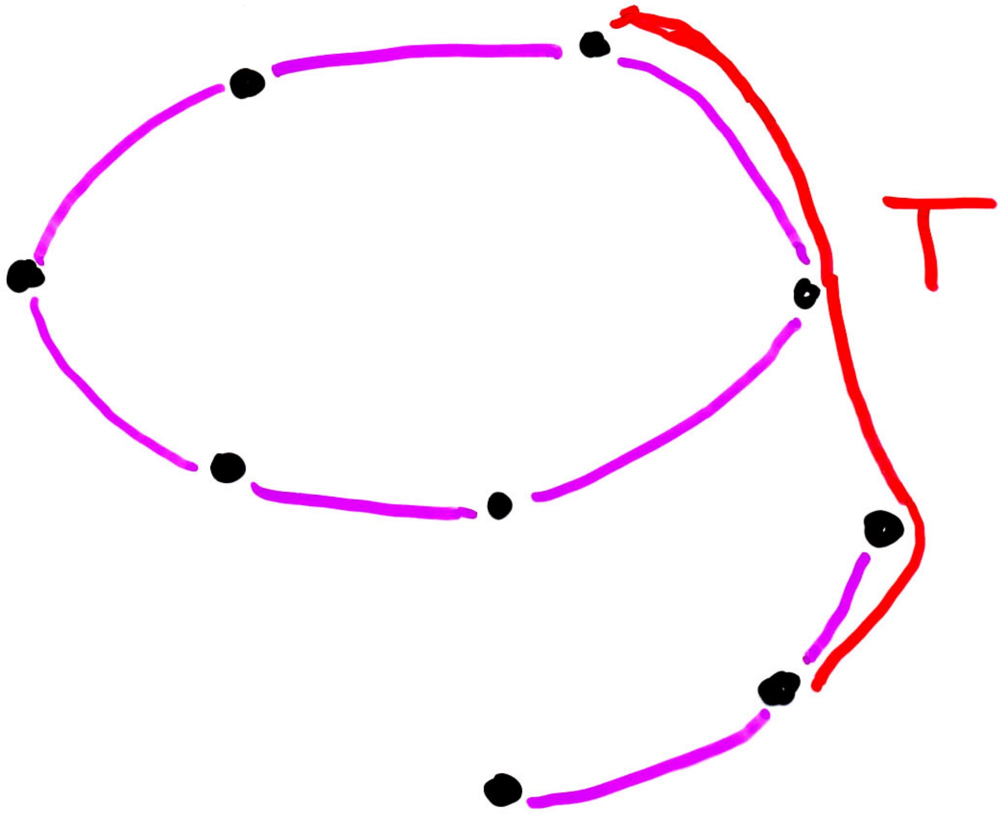
A GRAPH H HAS THE SAME NODES AS G.

D = THE SET OF EDGES WITH BOTH ENDS IN THE SAME CYCLE OF A

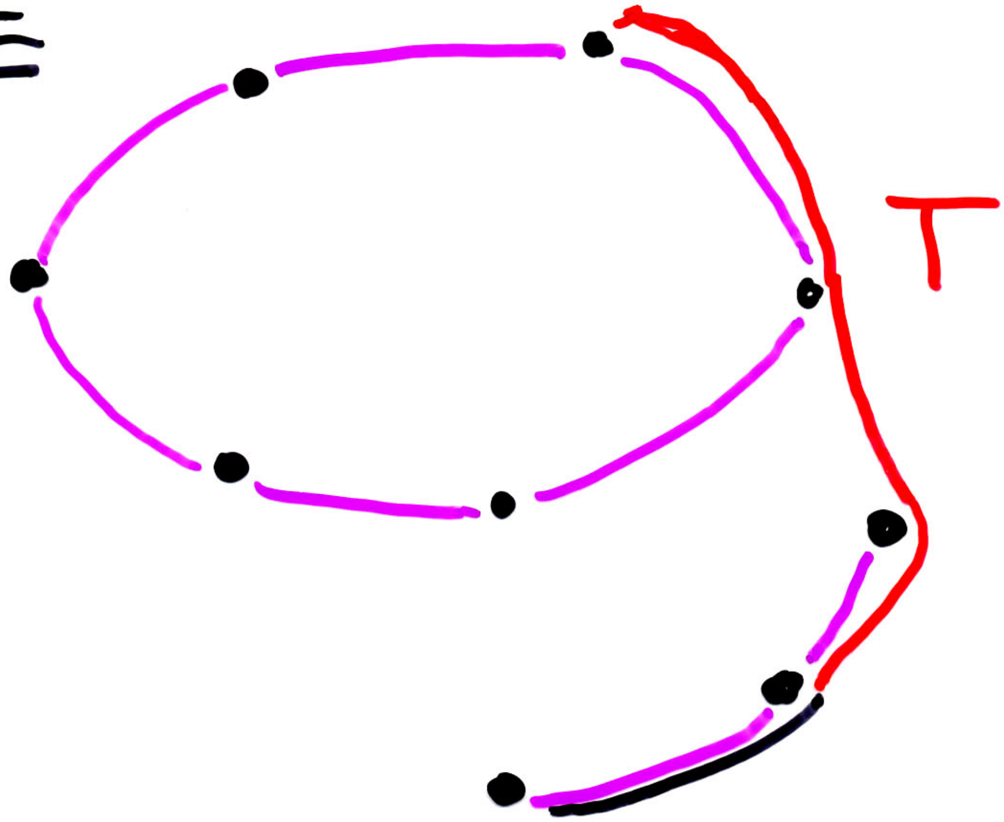


H HAS THE EDGE SET $\boxed{A \cup T - D}$ WITH THE FOLLOWING 3 COLOR CLASSES:

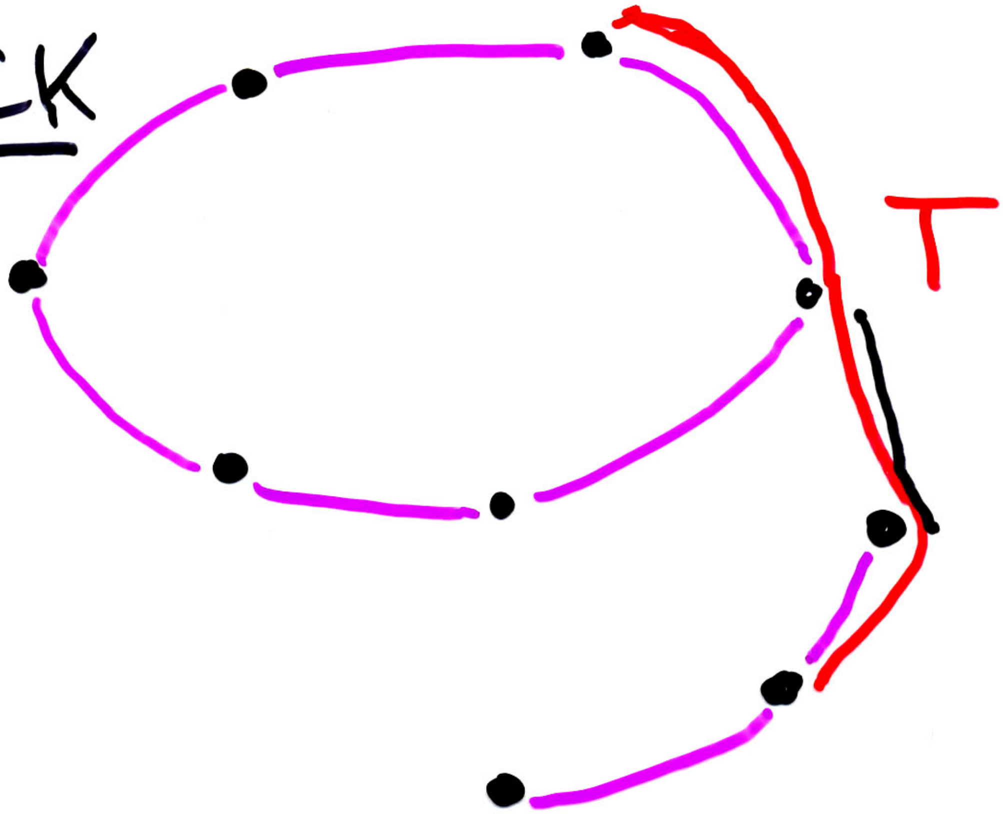
WHITE = $\boxed{A - T - D}$,
BLACK = $\boxed{T - A - D}$, AND
GREY = $\boxed{A \cap T - D}$.


T

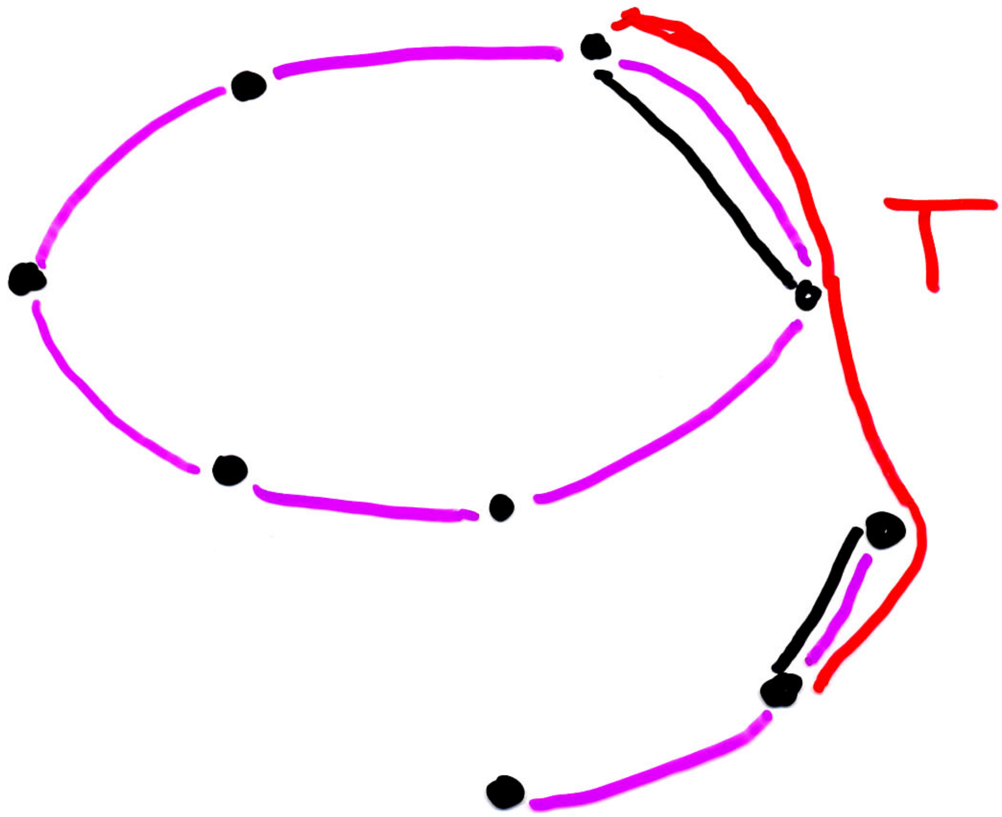
WHITE



BLACK

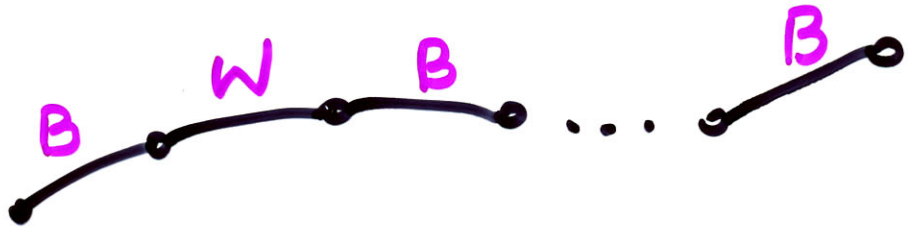


GREY



(21)

AN ALTERNATING PATH (AP) IS A PATH THAT STARTS AND ENDS WITH BLACK EDGE AND IN WHICH BLACK AND WHITE EDGES ALTERNATE.



AP

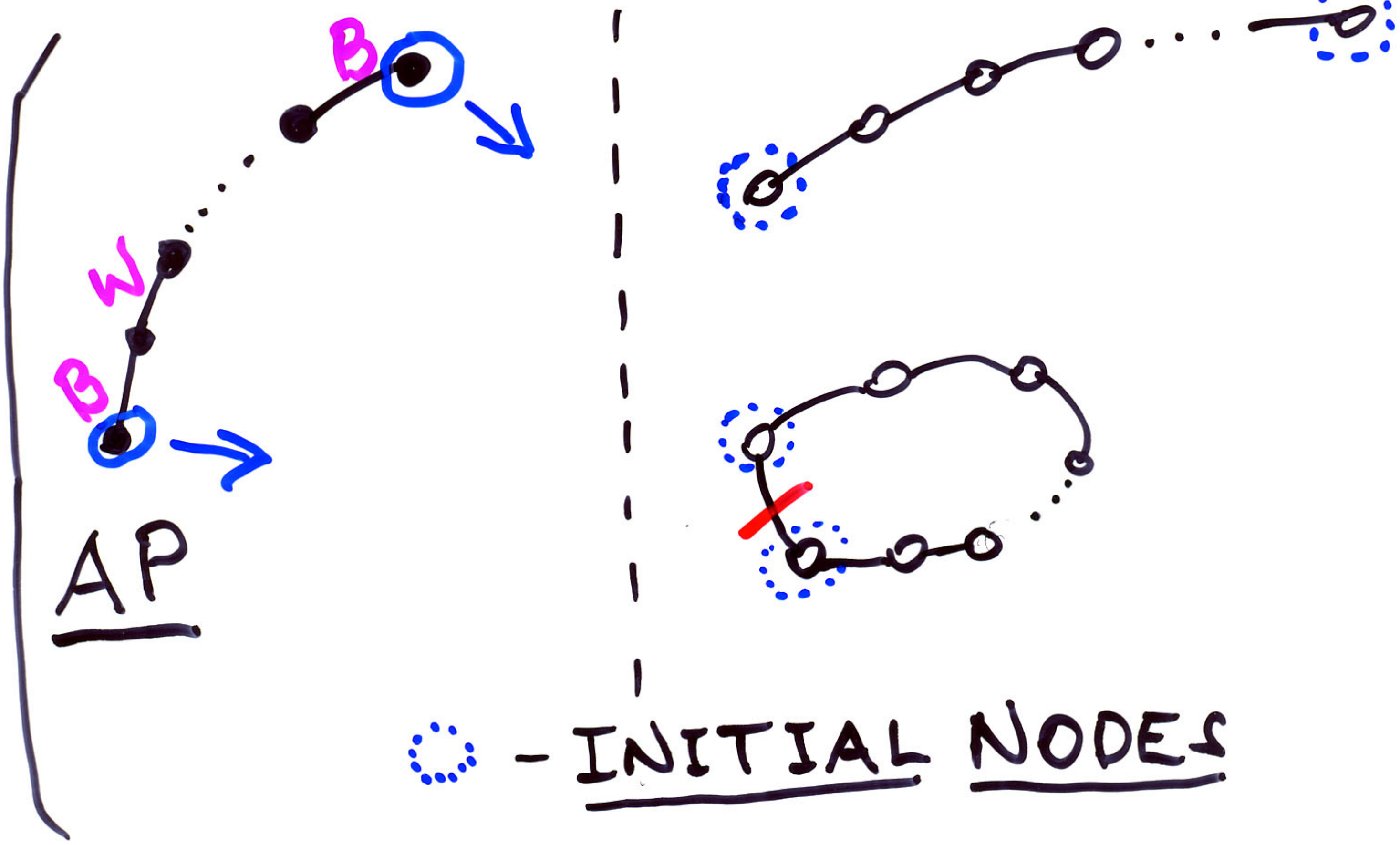
STARTING POINT:

ARBITRARY DECOMPOSITION OF WHITE AND BLACK EDGES INTO APs.

LATER, CONSECUTIVE ALTERNATION OF THAT DECOMPOSITION.

Analysis Idea:

Given a solution set A. Paths and cycles in A are called A-objects.



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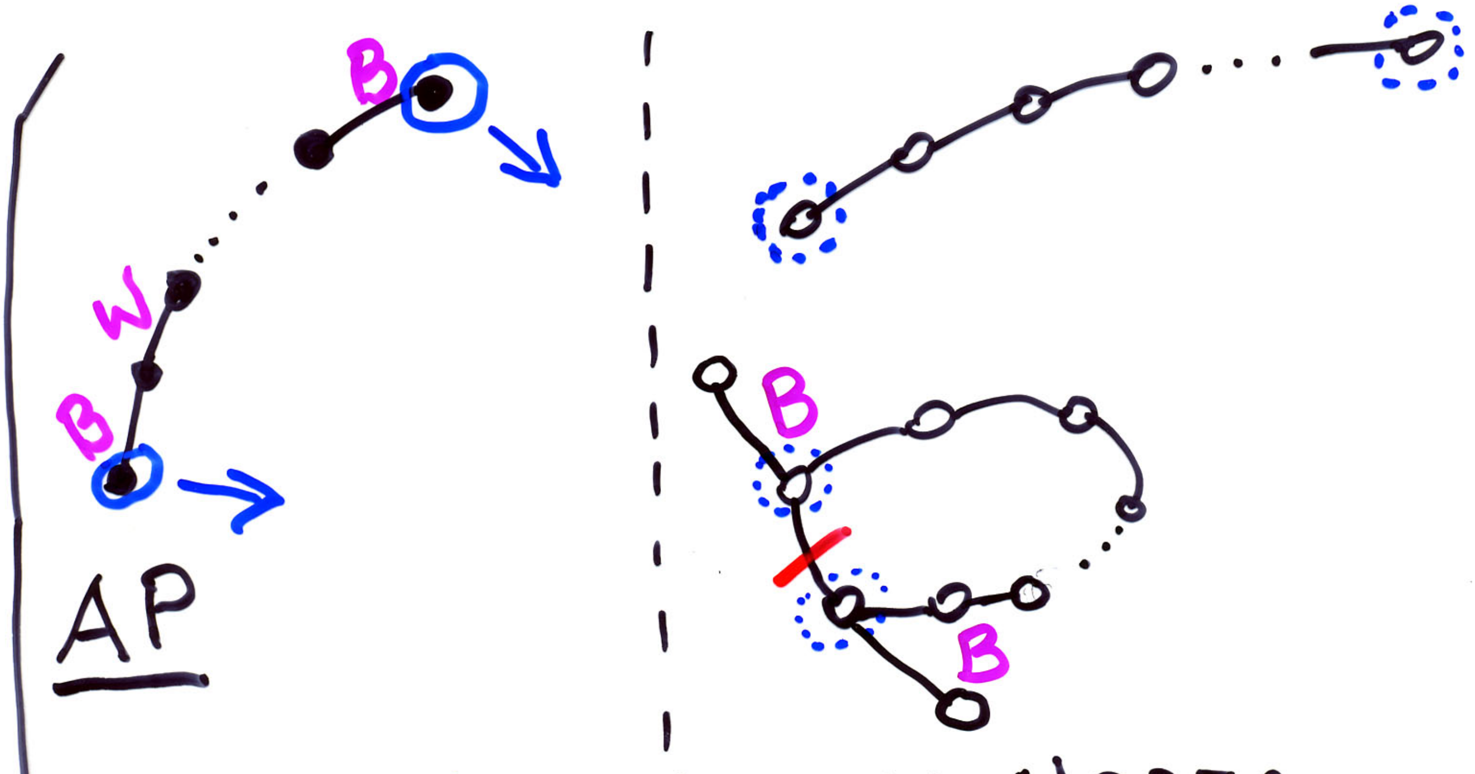

- Initial nodes

FACT. THERE ARE PAIRS OF INITIAL NODES IN A-CYCLES WITH LESS THAN 8 NODES.

AMORTIZED ANALYSIS GIVES:

$$\boxed{7k \leq n + 8k^*}$$

FOR A "GOOD ENOUGH" SOLUTION A, I.E. COST OF A $\leq 8/7\,(n + k^*)$

FURTHER RESEARCH:

1. IMPROVE RUNNING TIME ANALYSIS OF THE ALG. (REDUCE SEARCH SPACE FOR C's)

2. IMPROVE A.R.'s ($9/8$, $10/9$?)

MORE DETAILS IN:

P. BERMAN, M. KARPINSKI, 8/7-APPROXIMATION ALGORITHM FOR (1,2)-TSP, PROC. 17TH ACM-SIAM SODA (2006), 641–648.

FULL VERSION:

TR05-069, ECCC, 2005.
http://eccc.hpi-web.de/eccc-reports/

// APPROXIMATION HARDNESS PART:

L. ENGEBRETSEN, M. KARPINSKI, TSP WITH BOUNDED METRICS, RESEARCH REPORT No. 85273-CS, UNIV. BONN, 2006; http://theory.cs.uni-bonn.de/reports/.