Inapproximability of Dominating Set on Power Law Graphs (Revised Version)

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Abstract

We prove the first logarithmic lower bounds for the approximability of the Minimum Dominating Set problem for the case of connected (α, β)-power law graphs for α being a size parameter and β the power law exponent. We give also a best up to now upper approximation bound for this problem in the case of the parameters $\beta > 2$. We develop also a new functional method for proving lower approximation bounds and display a sharp approximation phase transition area between approximability and inapproximability of the underlying problems. Our results depend on a method which could be also of independent interest.

Keywords: Approximation Algorithms, Inapproximability, Power Law Graphs, Combinatorial Optimization, Dominating Set

1 Introduction

The Minimum Dominating Set problem (Min-DS) asks for a minimum size set of vertices D for a given graph G such that each vertex in G is either contained in D or adjacent to some vertex in D. The Min-DS problem has asymptotically the same approximation upper and lower bounds as the Set Cover problem. It can be approximated within $(1 - o(1)) \ln(n)$ by a greedy algorithm and, unless $\mathsf{NP} \subseteq \mathsf{DTIME}(n^{O(\log \log n)})$, there is no $(1 - \varepsilon) \ln(n)$-approximation algorithm for Min-DS for any $\varepsilon > 0$ [8]. Furthermore, Raz and Safra established an approximation lower bound of $c \cdot \ln(n)$ for some constant c under the weaker assumption that $\mathsf{P} \neq \mathsf{NP}$ [20].

In this paper we give new approximation upper and lower bounds for Min-DS on power law graphs. G is called a *power law graph* if the number of nodes of degree i is proportional to $i^{-\beta}$, for some $\beta > 0$. The parameter β is called the *power law exponent* and determines the log-log growth

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rate of G. The MIN-DS problem on power law graphs was originally introduced in the context of the sensor placement problems in massive social networks (cf. [7]).

Power law graphs (PLG) have been used in modeling and analyzing the real-world networks like the graphs of the Internet and the World Wide Web (WWW), peer-to-peer networks, mobile call networks, protein-protein interaction networks, gene regulatory networks, food webs and various social networks. Typically, the power law exponent of these real-world networks lies within the range $2 < \beta < 3$ (e.g. $\beta = 2.38$ for the WWW [5], $\beta = 2.4$ for protein-protein interaction networks [13]). There also exist examples of real-world networks with a power law exponent $\beta \leq 2$ or $\beta \geq 3$, e.g. for distributional food webs ($\beta = 1.05$, [18]), statistical investigations of book sales in the US ($\beta = 3.51$, [12, 19]) and human contact networks ($\beta = 3.4$, [17]).

A number of different random graph models were proposed in order to capture the topological properties of real-world networks and to analyze these graphs on the basis of a so called *null-model* (see [3, 15, 16, 1, 2, 6, 4]). On this basis, two different types of models have been introduced. The *evolving models* define a random process where one node at a time is added and connected to the existing graph in a random fashion—and thus are aiming to describe how power laws arise. The *static models* start from a given power law degree sequence as an input and then perform a random selection from the space of graphs with this degree sequence. The most prominent examples of the two types are the *preferential attachment model* described by Barabási [3], and the *ACL model* introduced by Aiello, Chung, and Lu [1, 2].

In this paper, we consider the power law model (α, β)-PLG due to Aiello, Chung, and Lu (also called the ACL model). A (multi-)graph G with maximum degree Δ is called an (α, β)-PLG with size parameter α and a power law exponent β, if for each $i \leq \Delta = \left\lfloor e^{\alpha/\beta} \right\rfloor$, the number of nodes of degree i is equal to $\left\lfloor e^{\alpha}/i^{\beta} \right\rfloor$.

2 Previous Results

Ferrante, Pandurangan, and Park [9] have shown the NP-hardness of MIN-DS on simple disconnected (α, β)-PLG for $\beta > 0$. In [21] it was shown that MIN-DS on (α, β)-PLG is in APX for $\beta > 2$. Furthermore, for $\beta > 1$, APX-hardness was shown and explicit constant approximation lower bounds were given, namely $1 + \frac{1}{390(2\zeta(\beta)3^{\beta}-1)}$ on (α, β)-PLG multigraphs and $1 + \frac{1}{3120\zeta(\beta)3^{\beta}}$ on simple (α, β)-PLG.

Eubank et al. [7] studied a relaxed version of MIN-DS: In the $(1-\varepsilon)$-MIN-DS problem the requirement is to dominate at least an $(1-\varepsilon)$-fraction of the vertices. They show that for every $\varepsilon > 0$, the $(1-\varepsilon)$-MIN-DS problem on bipartite random PLG admits a PTAS.

3 Our Results

In this paper, we give the first logarithmic lower approximation bounds for MIN-DS on (α, β)-PLG for the case $\beta \leq 2$. The best up to now approximation lower bound was a constant bound [21]. We show that in this case, unless $\mathsf{NP} \subseteq \mathsf{DTIME}(n^{O(\log \log n)})$, MIN-DS on connected (α, β)-PLG cannot be approximated within an approximation ratio $\Omega(\ln(n))$. Thus our lower approximation bound is almost tight. We also give improved approximation upper bounds for the case $\beta > 2$ and show that in this case, MIN-DS on (α, β)-PLG can be approximated within some constant approximation ratio R_β which converges to 1 as $\beta \to \infty$.

Then we take a very precise look at the phase transition point at $\beta = 2$. We consider a case when $\beta = 2 + 1/f(n)$ is a function of the size n of the graph. Here, n denotes the number of vertices of the PLG, and f is a monotone increasing unbounded function. This is an extension of the (α, β)-PLG model in [1], for which β was always a fixed constant. Surprisingly, we obtain a very sharp phase transition result, between approximability and inapproximability areas depending on the order of magnitude of the function f. We show that when $f(n) = o(\log n)$, MIN-DS on $(\alpha, 2 + 1/f(n))$-PLG is still in APX. On the other hand, we give a logarithmic approximation lower bound for the case when $f(n) = \omega(\log n)$.

Our approximation lower bounds are based on a direct approximate reduction from the SET COVER problem to the MIN-DS problem combined with an embedding of the resulting graph instances into (α, β)-PLG. Our constructions rely on precise estimates of sizes of node intervals in (α, β)-PLG and on the available node degree inside these intervals. Table 1 summarizes our main results in lower and upper approximation bounds for MIN-DS on (α, β)-PLG.

Power Law Exponent	Approx. Lower Bound
$0 < \beta < 1$	$\Omega\left(\ln(n) - \ln\left(\frac{1}{1-\beta}\right)\right)$
$\beta = 1$	$\Omega\left(\ln(n)\right)$
$1 < \beta < 2$	$\Omega\left(\ln(n) - \ln(\zeta(\beta))\right)$
$\beta = 2$	$\Omega\left(\ln(n) - \ln(\zeta(\beta))\right)$
$\beta = 2 + \frac{1}{f(n)}, f(n) = \omega(\log n)$	$\Omega\left(\ln(n) - \ln(\zeta(\beta))\right)$

Power Law Exponent	Approx. Upper Bound
$\beta = 2 + \frac{1}{f(n)}, f(n) = o(\log n)$	APX
$2 < \beta \leq 2.729$	$\frac{\zeta(\beta)-1}{\zeta(\beta)-\sum_{j=1}^{d-1} j^{-\beta}}$
$\beta > 2.729$	$\frac{\zeta(\beta-1)-2\zeta(\beta)}{\zeta(\beta-1)-2}$

Table 1: Summary of the main results: Approximation lower bounds and approximation upper bounds for MIN-DS on (α, β)-PLG for certain ranges of the parameter β. The precise choice of the parameter d is described in Theorem 4.

4 Organization of the Paper

In Section 5, we are giving an outline of the proof methods and the simulating constructions on which our reductions are based. In Section 6, we use the original reduction of Feige [8] from 5Occ-Max-E3-Sat (5 Occurrence Maximum E3-Sat) to the Set Cover problem and the reduction from the Set Cover to Min-DS. As a result of this section, we obtain sufficient information about the degree distribution of the resulting Min-DS instances $G_{U,\mathcal{S}}$. In Section 7 we give new lower bounds on the approximability of Min-DS on (α, β)-PLG. The case $0 < \beta < 1$ is treated in Section 7.1, based on a precise rounding error analysis for the terms that determine the lower approximation bound. A similar analysis is used for the case $\beta = 1$ in Section 7.2. The Section 7.3 deals with the case $1 < \beta \leq 2$. Especially, we describe how to *rescale* the degree distribution of instances $G_{U,\mathcal{S}}$ in order to embed them into an (α, β)-PLG. In Section 8 we present new upper bounds for the case of $\beta > 2$ and provide a detailed comparison of the previous and new upper bounds in terms of the parameter β. In Section 9 we consider the functional case when $\beta_f = 2+\frac{1}{f(n)}$ is a function of the graph size n which converges from above to 2.

5 Outline of the Method

We are going to give an outline of our methods and the underlying constructions. In order to obtain logarithmic approximation lower bounds for the Min-DS problem on (α, β)-*power law graphs*, we construct reductions from Min-DS in graphs, the problem which is basically as hard to approximate as the Set Cover problem. It is well known (cf. [14]) that Set Cover instances $(U, \mathcal{S})$ with universe U and set system $\mathcal{S}$ can be translated into instances $G_{U,\mathcal{S}}$ of Min-DS in graphs, where $G_{U,\mathcal{S}}$ contains a vertex for every element of U and vertices for the sets $S \in \mathcal{S}$. Element vertices are connected to set vertices of those sets in which they are contained, and two set vertices are connected by an edge if and only if the two sets have a non-empty intersection.

In our reduction we want to embed a Min-DS instance $G_{U,\mathcal{S}}$ into an (α, β)-PLG instance $\mathcal{G}_{\alpha,\beta}$ such as to carry over the logarithmic inapproximability from Min-DS to the special case of Min-DS in power law graphs. We will construct $\mathcal{G}_{\alpha,\beta}$ in such a way that it contains $G_{U,\mathcal{S}}$ as an induced subgraph and such that Min-DS$(\mathcal{G}_{\alpha,\beta}) = (1 + o(1))$Min-DS$(G_{U,\mathcal{S}})$. Here, Min-DS$(G)$ denotes the size of a minimum dominating set in the graph G. In particular $\mathcal{G}_{\alpha,\beta}$ will contain a set of vertices X which dominates all the vertices in $\mathcal{G}_{\alpha,\beta} \setminus G_{U,\mathcal{S}}$ and is of size $o($Min-DS$(G_{U,\mathcal{S}}))$.

In order to construct a power law graph $\mathcal{G}_{\alpha,\beta}$ for a given Min-DS instance $G_{U,\mathcal{S}}$, we need to know the degree distribution in the graph $G_{U,\mathcal{S}}$. In Section 6 we use the original construction from [8], and obtain upper and lower bounds for the degrees of nodes in the graph $G_{U,\mathcal{S}}$, where $(U, \mathcal{S})$ is a Set Cover instance. We apply our construction only to those Set Cover instances $(U, \mathcal{S}) = F_{SC}(\varphi)$ where φ is a 5Occ-Max-E3-Sat instance and F_{SC} is Feige's reduction from [8]. We show that the resulting Min-DS instances $G_{U,\mathcal{S}}$ have the following property: There exist constants $0 < a < b < 1$ such that for every $(U, \mathcal{S})$ with $(U, \mathcal{S}) = F_{SC}(\varphi)$, the node degrees of all vertices in $G_{U,\mathcal{S}}$ are contained in the interval $\left[N^a, N^b\right]$, where N is the number of vertices of the graph $G_{U,\mathcal{S}}$.

Intervals and Volumes. In our construction of a power law graph $\mathcal{G}_{\alpha,\beta}$ for a given graph $G_{U,\mathcal{S}}$, we choose X to be the set of all nodes in $\mathcal{G}_{\alpha,\beta}$ whose degree is within an interval $[x\Delta, y\Delta]$. Here, $\Delta = \lfloor e^{\alpha/\beta} \rfloor$ is the maximum degree of $\mathcal{G}_{\alpha,\beta}$, and $0 < x < y \le 1$ are parameters of our construction. In the subsequent sections we will show how to choose x and y for the different subcases ($0 < \beta < 1, \beta = 1$ and $1 < \beta \le 2$) of the power law distribution.

Let us now properly introduce the notions of node *intervals* and *volumes* (= sums of node degrees) of such vertex subsets. Let $\mathcal{G}_{\alpha,\beta} = (V, E)$ be an (α, β)-power law graph with n nodes. Thus, we have $n = \sum_{i=1}^{\Delta} \left\lfloor \frac{e^\alpha}{i^\beta} \right\rfloor$. Let $m = |E| = \frac{1}{2} \sum_{i=1}^{\Delta} i \left\lfloor \frac{e^\alpha}{i^\beta} \right\rfloor$ be the number of edges of $\mathcal{G}_{\alpha,\beta}$. According to [1, 2], the parameters n, m, α and β are related roughly as follows:

$$n \approx \begin{cases} \zeta(\beta)e^\alpha & \text{if } \beta > 1 \\ \alpha e^\alpha & \text{if } \beta = 1 \\ \frac{e^{\frac{\alpha}{\beta}}}{1-\beta} & \text{if } 0 < \beta < 1 \end{cases} \qquad \text{and} \qquad m \approx \begin{cases} \frac{1}{2}\zeta(\beta - 1)e^\alpha & \text{if } \beta > 2 \\ \frac{1}{4}\alpha e^\alpha & \text{if } \beta = 2 \\ \frac{1}{2}\frac{e^{\frac{2\alpha}{\beta}}}{2-\beta} & \text{if } 0 < \beta < 2 \end{cases}$$

An *interval* of nodes in $\mathcal{G}_{\alpha,\beta}$ is a set $[a, b] = \{v \in V \mid a \le \mathsf{deg}(v) \le b\}$, where $1 \le a \le b \le \Delta = \left\lfloor e^{\alpha/\beta} \right\rfloor$. Let $|[a, b]|$ be the number of nodes inside the interval $[a, b]$. The *volume* of an interval $[a, b]$ is defined as $\mathrm{vol}([a, b]) = \sum_{j=a}^{b} j \left\lfloor \frac{e^\alpha}{j^\beta} \right\rfloor$, the sum of node degrees of nodes inside the interval.

Embedding Technique. We will now describe in detail how to embed a given graph $G_{U,\mathcal{S}}$ into an (α, β)-PLG $\mathcal{G}_{\alpha,\beta}$. Let $G_{U,\mathcal{S}} = (V_{U,\mathcal{S}}, E_{U,\mathcal{S}})$ and let $N = |V_{U,\mathcal{S}}|$ be the number of vertices. As we will show in Section 6, the graphs $G_{U,\mathcal{S}}$ have the following property: There exist constants $0 < a < b < 1$ such that for all $v \in V_{U,\mathcal{S}}$, $N^a \le \mathsf{deg}_{U,\mathcal{S}}(v) \le N^b$, where $\mathsf{deg}_{U,\mathcal{S}}(v)$ is the degree of vertex v in the graph $G_{U,\mathcal{S}}$. The power law graph $\mathcal{G}_{\alpha,\beta} = (V_{\alpha,\beta}, E_{\alpha,\beta})$ has the vertex set $V_{\alpha,\beta} = V_{U,\mathcal{S}} \cup X \cup W \cup \Gamma$, with its components defined as follows:

- for $i = 1, 2$, V_i is the set of degree i nodes in $\mathcal{G}_{\alpha,\beta}$,
- $\Gamma \subseteq V_2$ is the set of neighbors of $V_{U,\mathcal{S}}$ in $\mathcal{G}_{\alpha,\beta} \setminus G_{U,\mathcal{S}}$,
- X is a set of size $|X| = o(\textsc{Min-DS}(G_{U,\mathcal{S}}))$ which dominates all the vertices in $\mathcal{G}_{\alpha,\beta} \setminus G_{U,\mathcal{S}}$,
- W is a set of nodes which are used in the implementation of the power law distribution.

The set Γ is to enforce any *reasonable* dominating set in $\mathcal{G}_{\alpha,\beta}$ to contain a dominating set of the graph $G_{U,\mathcal{S}}$. This means nodes in $G_{U,\mathcal{S}}$ will be dominated by nodes from $G_{U,\mathcal{S}}$ and not by nodes from $\mathcal{G}_{\alpha,\beta} \setminus G_{U,\mathcal{S}}$.

The set X is a subset of the node interval $[x\Delta, y\Delta] = \{ v \in V_{\alpha,\beta} \mid x\Delta \leq \mathsf{deg}_{\alpha,\beta}(v) \leq y\Delta \}$ for some parameters $0 < x < y \leq 1$. W is the set of remaining nodes which serves to implement the power law distribution. The power law graph $\mathcal{G}_{\alpha,\beta}$ is constructed in such a way that each node in $V_{U,\mathcal{S}}$ has precisely one neighbor in $\Gamma \subseteq V_2$, and every $u \in \Gamma$ has precisely one neighbor in $V_{U,\mathcal{S}}$. Furthermore, each node $w \in W$ is adjacent to at least one node in X, whereas each $v \in X$ has at least one degree 1 neighbor in W. In order to guarantee that the graph $\mathcal{G}_{\alpha,\beta}$ is connected, we include all the edges of a spanning tree on X. From this construction it follows that the set X dominates every vertex in $W \cup \Gamma$ (cf. Figure 1).

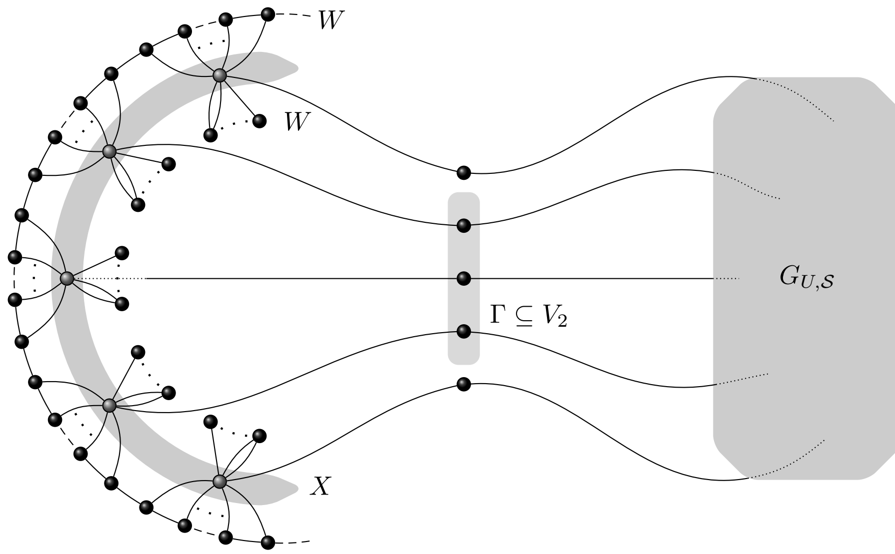

Figure 1: The main construction for the embedding of a Min-DS (Set Cover) instance $G_{U,\mathcal{S}}$ into a (α, β)-PLG. In the resulting graph the nodes $\bullet \in X$ are dominating the sets $W \cup V_1$, separating the dominating set in $G_{U,\mathcal{S}}$ from the dominating set in $\mathcal{G}_{\alpha,\beta} \setminus G_{U,\mathcal{S}}$.

The graph $\mathcal{G}_{\alpha,\beta}$ will be constructed as follows. First the set of nodes $V_{\alpha,\beta} = V_{U,\mathcal{S}} \cup X \cup W \cup \Gamma$ is generated. To each node $v \in V_{\alpha,\beta}$ we assign a value $\mathsf{deg}_{\alpha,\beta}(v)$, its node degree in the graph $\mathcal{G}_{\alpha,\beta}$. Then we generate the edges of the graph. During the construction, we keep track of residual degrees $\mathsf{deg}_r(v)$ of nodes $v \in V$, where in each step of the construction, $\mathsf{deg}_r(v)$ is equal to $\mathsf{deg}_{\alpha,\beta}(v)$ minus the current number of edges incident to v.

The algorithm `ConstructPLG` (Figure 2 on page 7) gets as an input the graph $G_{U,\mathcal{S}}$ for a given Set Cover instance $(U, \mathcal{S})$ and constructs the associated power law graph $\mathcal{G}_{\alpha,\beta}$. The procedure `Fill_Wheel` (Figure 3 on page 8) gets as an input a set of nodes V with residual degrees $\mathsf{deg}_r(v) > 0, v \in V$ and generates the missing edges degree-wise in a cyclic order. Let $v_{j,1}, \ldots, v_{j,n_j}$ be the nodes of degree $\mathsf{deg}_r(v_{j,l}) = j$ in the set V, then the following invariant will be maintained. In every stage of the construction, for every $j \in \{1, \ldots, \Delta\}, \mathsf{deg}_r(v_{j,1}) \leq \cdots \leq \mathsf{deg}_r(v_{j,n_j})$ and $\mathsf{deg}_r(v_{j,n_j}) - \mathsf{deg}_r(v_{j,1}) \leq 1$. A previous version of the procedure `Fill_Wheel` has been used in [11]. Figure 4 shows how the node intervals with the same residual degree are filled and how

Algorithm 1: `ConstructPLG`

Input: $G_{U,\mathcal{S}} = (V_{U,\mathcal{S}}, E_{U,\mathcal{S}})$ with $|V_{U,\mathcal{S}}| = N$.

Output: Power law graph $\mathcal{G}_{\alpha,\beta} = (V_{\alpha,\beta}, E_{\alpha,\beta})$ with $V_{\alpha,\beta} = V_{U,\mathcal{S}} \cup X \cup W \cup \Gamma$, $|V_{\alpha,\beta}| = n$ and $E_{U,\mathcal{S}} \subseteq E_{\alpha,\beta}$.

choose parameters α, x, y such that $\mathrm{vol}([x\Delta, y\Delta]) \geq n$ and $|[N^a, N^b]| \geq N$;
`/* In the subsequent sections we will describe in detail how to choose the parameters` α, x, y `for the various subcases of` β `*/`
let $V_{U,\mathcal{S}} = \{u_1, \ldots, u_N\}$;
set $V_{\alpha,\beta} := V_{U,\mathcal{S}} \cup X \cup W \cup \Gamma$, where $X = [x\Delta, y\Delta]$ and $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$;
assign degrees $\mathsf{deg}_{\alpha,\beta}(v), v \in V_{\alpha,\beta}$:
  Nodes $u_i \in V_{U,\mathcal{S}}$ get node degree $\mathsf{deg}_{\alpha,\beta}(u_i) = \mathsf{deg}_{U,\mathcal{S}}(u_i) + 1$;
  Nodes $\gamma_i \in \Gamma$ get node degree $\mathsf{deg}_{\alpha,\beta}(\gamma_i) = 2$;
  For each $x\Delta \leq j \leq y\Delta, X$ contains $\left\lfloor \frac{e^\alpha}{j^\beta} \right\rfloor$ nodes $x_{j,l}$ of node degree $\mathsf{deg}_{\alpha,\beta}(x_{j,l}) = j$;
  W contains the remaining nodes from $V_{\alpha,\beta}$, with node degrees assigned accordingly;
for each $v \in V_{\alpha,\beta}$, set $\mathsf{deg}_r(v) := \mathsf{deg}_{\alpha,\beta}(v)$;
add matching $M := \{\{\gamma_i, u_i\} | i \leq N\}$ to $E_{\alpha,\beta}$, decrease residual degrees in $\Gamma \cup V_{U,\mathcal{S}}$ by 1;
add edges of a spanning tree for X to $E_{\alpha,\beta}$, update residual degree of nodes in X;
for each $x_{i,j} \in X$ add an edge to some node $w \in V_1$ of nonzero residual degree;
for each $v \in V_{\alpha,\beta} \setminus (X \cup V_{U,\mathcal{S}})$
  add an edge from v to some node $x_{i,j} \in X$ of nonzero residual degree;
  update $\mathsf{deg}_r(v), \mathsf{deg}_r(x_{i,j})$;
`Fill_Wheel`$(W \cup X)$; `/* realizes residual degrees on` W `and` X `*/`
return $\mathcal{G}_{\alpha,\beta} = (V_{\alpha,\beta}, E_{\alpha,\beta})$;

Figure 2: The algorithm `ConstructPLG` constructs the power law graph.

the case is treated when the number of nodes contained in such an interval is odd. The set $X \subseteq [x\Delta, y\Delta]$ and the parameters $x\Delta$ and $y\Delta$ of the construction are chosen such that the volume $\mathrm{vol}([x\Delta, y\Delta]) = \sum_{j=x\Delta}^{\Delta} \left\lfloor \frac{e^\alpha}{j^\beta} \right\rfloor \cdot j$ minimally exceeds the number of nodes in $V_{\alpha,\beta} \setminus X$. Thus, some nodes $v \in X$ might have residual degree > 0. In this case, `Fill_Wheel`(X) is used to reduce the residual degree of these nodes to 0. Furthermore, each node $w \in W$ is connected to the set X by a single edge. Since the residual degrees of nodes $w \in W$ are within the interval $[3, \Delta]$, `Fill_Wheel`(W) is used to reduce the residual degree of nodes $w \in W$ to 0.

In the subsequent sections, we will show how to choose the parameters x and y of the construction depending on the power law exponent β in such a way that the set X becomes sufficiently small. It follows directly from the construction that for each dominating set D in $G_{U,\mathcal{S}}$, the set $X \cup D$ is a dominating set in $\mathcal{G}_{\alpha,\beta}$. Moreover, for $\beta \leq 2$ we will be able to choose x and y such that $|X| = o(\mathrm{OPT}(G_{U,\mathcal{S}}))$. Since $\mathrm{OPT}(G_{U,\mathcal{S}})$ is a lower bound for the size of a dominating set in $\mathcal{G}_{\alpha,\beta}$, this will yield a logarithmic approximation lower bound for MIN-DS on (α, β)-PLG.

Algorithm 2: `Fill_Wheel`

Input: Set of nodes V with residual degrees $\mathsf{deg}_r(v) \geq 0$, $v \in V$.
Output: A multigraph $W = (V, E)$ such that each node $v \in V$ has degree $\mathsf{deg}_r(v)$ in W.

for $j = 1, \ldots, \Delta$ let $v_{j,1}, \ldots, v_{j,n_j}$ be the nodes of degree $\mathsf{deg}_r(v_{j,l}) = j$ in V;
/* We may assume that both n_1 and $\sum_{i \geq 2} i \cdot n_i$ are even */
generate a matching on $v_{1,1}, \ldots, v_{1,n_1}$ and update residual degrees accordingly;
for $j = 2, \ldots, \Delta$ **do**
    while $\mathsf{deg}_r(v_{j,n_j}) > 0$ **do**
        choose l min such that $\mathsf{deg}_r(v_{j,l})$ is max;
        if $l < n_j$ **then**
            generate edge $\{v_{j,l}, v_{j,l+1}\}$;
            $\mathsf{deg}_r(v_{j,l}) := \mathsf{deg}_r(v_{j,l}) - 1$;
            $\mathsf{deg}_r(v_{j,l+1}) := \mathsf{deg}_r(v_{j,l+1}) - 1$;
        else if $l = n_j, \mathsf{deg}_r(v_{j,1}) > 0$ **then**
            generate edge $\{v_{j,l}, v_{j,1}\}$;
            $\mathsf{deg}_r(v_{j,l}) := \mathsf{deg}_r(v_{j,l}) - 1$;
            $\mathsf{deg}_r(v_{j,1}) := \mathsf{deg}_r(v_{j,1}) - 1$;
        else if $l = n_j, \mathsf{deg}_r(v_{j,1}) = 0, j < \Delta$ **then**
            generate edge $\{v_{j,l}, v_{j+1,1}\}$;
            $\mathsf{deg}_r(v_{j,l}) := \mathsf{deg}_r(v_{j,l}) - 1$;
            $\mathsf{deg}_r(v_{j+1,1}) := \mathsf{deg}_r(v_{j+1,1}) - 1$;

Figure 3: The procedure `Fill_Wheel` reduces the residual degrees of a node set to 0.

6 Node Degrees and Lower Bound for Set Cover

In order to go on with our proof we need the following constructions. We start with Feige's [8] logarithmic lower bound for the approximability of the Set Cover problem. For each Set Cover instance $(U, \mathcal{S})$ we embed the associated Min-DS instance $G_{U,\mathcal{S}}$ into an (α, β)-PLG $\mathcal{G}_{\alpha,\beta}$. In order to implement the power law node-degree distribution, we need to know the degree distribution of the graph $G_{U,\mathcal{S}}$. Therefore we briefly review the construction from [8]. This construction is based on a k-prover proof system for the problem 5Occ-Max-E3-Sat. Consider a 3CNF formula φ with n variables such that each variable occurs at most 5 times in φ. One can assume that either the formula is satisfiable, or no assignment satisfies more than an ε fraction of the clauses simultaneously. The k-prover proof system works as follows: It chooses k codewords of length $l = \Theta(\log \log n)$, weight $l/2$ and pairwise *Hamming distance* $\geq l/3$. The verifier picks l clauses $C_1, \ldots, C_l$ from φ independently uniformly at random. Independently, from each such clause C_i it picks one variable x_i of C_i uniformly at random. For each $1 \leq i \leq k$, the verifier sends to the prover i those $l/2$ clauses C_j for which the associated bit of prover i's codeword is 1 and those $l/2$ variables x_j for which the associated bit of prover i's codeword is 0. The provers return their answers, and

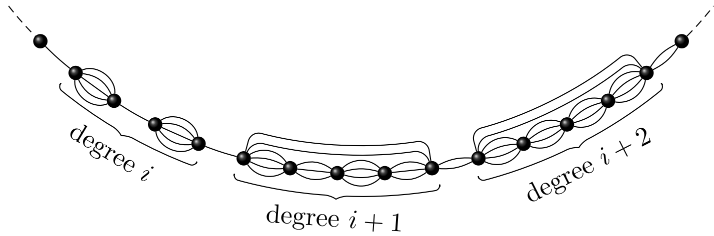

Figure 4: Procedure `Fill_Wheel` realizes the residual degrees on the wheel nodes in W and X.

based on this the verifier determines its output. The construction of the associated Set Cover instances makes use of some combinatorial building blocks called *partition systems*.

Following [8], we define a partition system $B(m, L, k, d)$ to consist of a ground set B of cardinality $|B| = m$ and L partitions $p_1, \dots, p_L$ of B into k disjoint subsets $p_{j,h} \subset B$. The defining property of these partition systems is that each cover of B by subsets $p_{j,h}$ which uses sets from pairwise different partitions must consist of at least d subsets. [8] gives a randomized construction of such partition systems with $L \approx (\log m)^c$, k being any number smaller than $\ln(m/3) \cdot \ln\ln(m)$ and $d = (1 - f(k)) \cdot k \cdot \ln(m)$ with some function $f(k)$ with $f(k) \longrightarrow 0$ as $k \longrightarrow \infty$. That construction yields partitions for which with high probability all the sets have the same size. We show that the same result is obtained by making use of random permutations. But now, for each partition p_j, the sets $p_{j,h}$ always have the same size m/k (provided $k|m$). Namely, choose a random permutation $\pi_j \in_R S_m$ and let $p_{j,h} = \{\pi_j((h-1)m/k+1), \dots, \pi_j(k \cdot m/k)\}$. Suppose now we cover B with d subsets $p_{j_1,h_1}, \dots, p_{j_d,h_d}$ from pairwise different partitions. Then for a given point $v \in B$, the probability that v is covered by at least one of them is

$$
\begin{aligned}
&P(\text{point } v \in B \text{ is covered by at least one of these } d \text{ sets})\\
&= 1 - \prod_{i=1}^{d} P\left(v \text{ is not in position } 1, \dots, m/k \text{ in permutation } \pi_j\right) = 1 - \left(\frac{\binom{m-1}{m/k} \cdot (\frac{m}{k})! \cdot (m - \frac{m}{k})!}{m!}\right)^d\\
&= 1 - \left(\frac{(m-1)! \cdot (m - \frac{m}{k})!}{(m - 1 - \frac{m}{k})! \cdot m!}\right)^d = 1 - \left(\frac{m \cdot \left(1 - \frac{1}{k}\right)}{m}\right)^d \;=\; 1 - \left(1 - \frac{1}{k}\right)^d \;.
\end{aligned}
$$

This is precisely the property of the randomized construction which has been used in [8] in the analysis of the construction. So from now on we assume that all sets of a partition p_j have the same size m/k.

Resulting Set Cover Instances ([8]). For a given 5Occ-Max-E3-Sat formula φ with n variables and the property that either φ is satisfiable or no assignment satisfies more than an ε fraction of the clauses, a Set Cover instance $(U, \mathcal{S})$ is constructed as follows:

- $\mathcal{R}$ is the set of random strings used by the verifier in the k-Prover Proof System. The number of random strings is $|\mathcal{R}| = R = (5n)^l$.

- $|U| = mR$ with $m = (5n)^{\frac{2l}{\varepsilon}}$, hence $|U| = (5n)^{l(1+2/\varepsilon)}$
- For each $r \in \mathcal{R}$, $B_r(m, L, k, d)$ is a partition system with $L = 2^l$.
- $Q = n^{l/2} \cdot \left(\frac{5n}{3}\right)^{l/2}$ is the number of different queries the verifier may ask to a prover.
- $\mathcal{S}$ contains for every triple (q, a, i) a set $S_{q,a,i}$, where q is a query, i is (the index of) a prover and a is the prover's answer. The set $S_{q,a,i}$ is defined as $S_{q,a,i} = \bigcup_{r\colon (q,i)\in r} B(r, a_r, i)$.

Hence the number of sets in $\mathcal{S}$ is $Q \cdot k$, and each set is of cardinality $\sqrt{R} \cdot m/k$. We have to give an estimate for the number of sets in which a point (an element of U) occurs. For each prover i, for each query q, each point in B_r with $|B_r| = m$ occurs in 2^l sets $S_{q,a,i}$. Hence the total degree of points (the number of occurrences of this point in sets) is $2^l \cdot Q$.

From Set Cover to Dominating Set. Let $(U, \mathcal{S})$ denote a Set Cover instance with $U = \{u_1, \ldots, u_{|U|}\}$ and $\mathcal{S} = \left\{S_1, \ldots, S_{|\mathcal{S}|}\right\}$. Let $G_{U,\mathcal{S}}$ be the undirected graph with set of vertices $V_{U,\mathcal{S}} = U \cup \mathcal{S}$ and set of edges $E_{U,\mathcal{S}} = \{\,\{S_i, u_j\} \mid u_j \in S_i\,\} \cup \{\,\{S_j, S_l\} \mid S_j \cap S_l \neq \emptyset\,\}$. We observe that each set cover $\mathcal{C} \subseteq \mathcal{S}$ is a dominating set in $G_{U,\mathcal{S}}$. On the other hand, let $D \subseteq V_{U,\mathcal{S}}$ be a dominating set in $G_{U,\mathcal{S}}$ with $D = D_U \cup D_{\mathcal{S}}, D_U = D \cap U$ and $D_{\mathcal{S}} = D \cap \mathcal{S}$. If we replace each $u_i \in D_U$ by an arbitrary set S_j with $u_i \in S_j$, the resulting set D' is a dominating set with $D_{\mathcal{S}} \subseteq D' \subseteq \mathcal{S}$ and $|D'| \leq |D|$. In this way dominating sets in $G_{U,\mathcal{S}}$ correspond to set covers $\mathcal{C}$ for $(U, \mathcal{S})$.

In the construction in [8], the parameter l satisfies $l = \Theta(\log\log n)$. If $N_0 = |U| + |\mathcal{S}|$ is the number of nodes of $G_{U,\mathcal{S}}$, then (up to logarithmic factors), $N_0 \approx n^l + n^{l(1+2/\varepsilon)}$, the degree of element nodes $u \in U$ is $\approx n^l$, each set contains $n^{l(1/2+2/\varepsilon)}$ elements and there are $\approx n^l$ sets. The degree of set nodes in $G_{U,\mathcal{S}}$ is bounded by the sum of the cardinality of that set and the number of sets in the instance $(U, \mathcal{S})$, which is $\approx n^{l(1/2+2/\varepsilon)}$. Hence we obtain the following result we will use in the sequel.

Lemma 1. *Let F_{SC} denote Feige's reduction from* 5Occ-Max-E3-Sat *to the* Set Cover *problem, and for a given* Set Cover *instance $(U, \mathcal{S}) = F_{SC}(\varphi)$ let $G_{U,\mathcal{S}}$ be the associated* Min-DS *instance as described above. If N_0 is the number of nodes of $G_{U,\mathcal{S}}$, then for every node v in $G_{U,\mathcal{S}}$, the node degree of v in $G_{U,\mathcal{S}}$ satisfies $N_0^a \leq \mathsf{deg}_{U,\mathcal{S}}(v) \leq N_0^b$, where $0 < a < b < 1$ and $b = (1+o(1)) \cdot \frac{1/2+2/\varepsilon}{1+2/\varepsilon} = (1 + o(1)) \cdot \frac{\varepsilon+4}{2\varepsilon+4}$.*

In the next section we consider approximation lower bounds for the values of β satisfying $0 < \beta \leq 2$.

7 New Lower Bounds for $0 < \beta \leq 2$

We will now describe our new logarithmic lower bounds for approximability of the Min-DS problem in (α, β)-PLG. We distinguish several cases depending on the range of the parameter β. For the cases $0 < \beta < 1$, $1 < \beta < 2$ and $\beta = 2$ our construction involves *rescaling* of the instances $G_{U,\mathcal{S}}$, which has the effect of shifting the degree interval $\left[N^a, N^b\right]$ towards the left end of the full interval $[1, \Delta]$. It turns out that for the case $\beta = 1$ we can omit the scaling and directly implement the power law distribution.

Bounds on Optimums in $G_{U,\mathcal{S}}$. Let $(U, \mathcal{S})$ be an instance of the Set Cover problem which is an image $(U, \mathcal{S}) = F_{SC}(\varphi)$ of some 5Occ-Max-E3-Sat instance φ under Feige's reduction F_{SC}. Let $|U| = mR$ be the number of points in $(U, \mathcal{S})$. It is shown in [8] that if the formula φ is satisfiable, then the set U can be covered by kQ subsets. If only a $(1-\varepsilon)$ fraction of the clauses in φ can be satisfied simultaneously, then at least $(1-\varepsilon)kQ \ln m$ subsets are needed in order to cover U.

Suppose now that the number of nodes of $G_{U,\mathcal{S}}$ is $N_0 = |U| + |\mathcal{S}|$. The number of points in the Set Cover instance is $|U| = mR = (5n)^{l(1+2/\varepsilon)}$. The number of sets is $|\mathcal{S}| = kQ$, where $Q = \left(\frac{5}{3}^{l/2} \cdot n^l\right) = |U|^c$ with a constant $c \sim \frac{\varepsilon}{2+\varepsilon}$. Furthermore, $R = (5n)^l = |U|^{c_m}$ with a constant $c_m = \frac{\varepsilon}{2+\varepsilon}$. Let $\mathsf{OPT}(G_{U,\mathcal{S}})$ denote a minimum cardinality dominating set of $G_{U,\mathcal{S}}$. We may assume that $\mathsf{OPT}(G_{U,\mathcal{S}})$ consists of nodes corresponding to sets of $\mathcal{S}$. Thus we obtain the following bounds:

- If the formula φ which we are starting from is satisfiable, then

$$|\mathsf{OPT}(G_{U,\mathcal{S}})| \leq k \cdot N_0^{\frac{\varepsilon}{2+\varepsilon}}$$

- If only a $(1-\varepsilon)$ fraction of the clauses in φ can be satisfied simultaneously, then

$$|\mathsf{OPT}(G_{U,\mathcal{S}})| \geq (1-\varepsilon) \cdot k \cdot N_0^{\frac{\varepsilon}{2+\varepsilon}} \cdot \frac{\varepsilon}{2+\varepsilon} \cdot \left(\frac{1}{2}\right)^{\frac{\varepsilon}{2+\varepsilon}} \cdot (\ln(N_0) - O(1))$$

Here k is the number of provers in Feige's k-prover proof system. Furthermore, as a result of Lemma 1, the node degrees in $G_{U,\mathcal{S}}$ are contained in the interval $\left[N_0^a, N_0^b\right]$ with $0 < a < b < 1$ being constant.

Scaling. We want to construct the power law graph $\mathcal{G}_{\alpha,\beta}$ with vertex set $V_{\alpha,\beta} = V_{U,\mathcal{S}} \cup X \cup W \cup \Gamma$ in such a way that the two node intervals containing the sets $V_{U,\mathcal{S}}$ and X are disjoint. This means that $V_{U,\mathcal{S}} \subseteq \left[N_0^a, N_0^b\right]$ and $X = [x\Delta, y\Delta]$ in $\mathcal{G}_{\alpha,\beta}$ such that $N^b < x\Delta$. Here N_0 denotes the cardinality of $V_{U,\mathcal{S}}$. Furthermore we want to guarantee that the size of X is $o(\mathsf{OPT}(G_{U,\mathcal{S}}))$.

In the cases $0 < \beta < 1$ and $1 < \beta \leq 2$ we are currently not able to achieve this directly. Therefore we start by rescaling the node degree distribution of the graph $G_{U,\mathcal{S}}$ appropriately before embedding it into the power law graph $\mathcal{G}_{\alpha,\beta}$. More precisely, we replace $G_{U,\mathcal{S}}$ by the graph $G^d_{U,\mathcal{S}}$ which consists of N_0^{d-1} disjoint copies of $G_{U,\mathcal{S}}$, where d is a parameter of the construction (cf. Figure 5). Let $N = N_0^d$ be the size of $G_{U,\mathcal{S}}^d$. The effect of scaling is two-fold: The node degree distribution is shifted from the interval $\left[N_0^a, N_0^b\right]$ to the interval $\left[[N^{a/d}, N^{b/d}\right]$. On the other hand, the size of an optimum dominating set is shifted from the interval $\left[N_0^{\frac{\varepsilon}{2+\varepsilon}}, N_0^{\frac{\varepsilon}{2+\varepsilon}} \cdot \log(N_0)\right]$ to $\left[N^{1-\delta}, N^{1-\delta} \cdot \log(N)\right]$, where we can choose δ arbitrary small by choosing d appropriately.

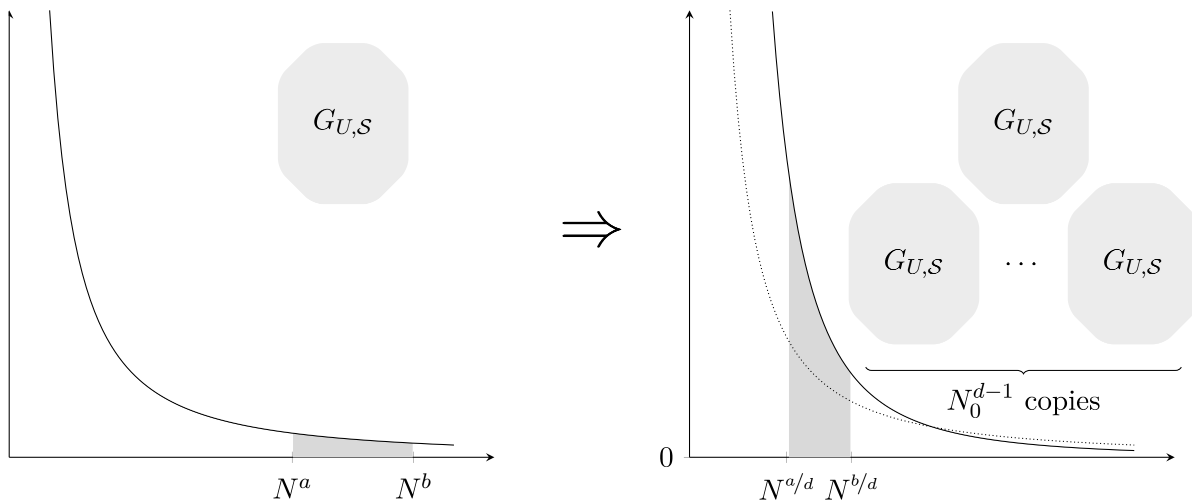

Figure 5: Scaling by replacing the original graph $G_{U,\mathcal{S}}$ by N_0^{d-1} disjoint copies in order to shift the occupied degree set towards the left end of the full interval.

Let $\mathsf{OPT}(G_{U,\mathcal{S}}^d)$ denote an optimum dominating set of $G_{U,\mathcal{S}}^d$. Using the bound for sizes of optimum dominating sets in $G_{U,\mathcal{S}}$ which we described above, we obtain the following bounds for the scaled graph $G_{U,\mathcal{S}}^d$:

- If the formula φ which we are starting from is satisfiable,

$$|\mathsf{OPT}(G_{U,\mathcal{S}}^d)| \leq N^{\frac{d-1}{d}} k N^{\frac{1}{d}\frac{\varepsilon}{2+\varepsilon}} = kN^{\frac{1}{d}\left(d-1+\frac{\varepsilon}{2+\varepsilon}\right)} \tag{7.1}$$

- If only a $(1-\varepsilon)$ fraction of the clauses in φ can be satisfied simultaneously, then

$$\begin{aligned} |\mathsf{OPT}(G_{U,\mathcal{S}}^d)| &\geq (1-\varepsilon) k N^{\frac{1}{d}\frac{\varepsilon}{2+\varepsilon}} \frac{\varepsilon}{2+\varepsilon} \left(\frac{1}{2}\right)^{\frac{\varepsilon}{2+\varepsilon}} \left(\ln\left(N^{\frac{1}{d}}\right) - O(1)\right) N^{\frac{d-1}{d}} \\ &= kN^{\frac{1}{d}\left(d-1+\frac{\varepsilon}{2+\varepsilon}\right)} \frac{\varepsilon(1-\varepsilon)}{2+\varepsilon} \left(\frac{1}{2}\right)^{\frac{\varepsilon}{2+\varepsilon}} \left(\ln\left(N^{\frac{1}{d}}\right) - O(1)\right) \end{aligned} \tag{7.2}$$

Choice of the Parameters of our Construction. Now we describe in detail how to choose the parameters α, x, y and the scaling parameter d of our construction. Recall that we start from a Set Cover instance $(U, \mathcal{S})$ and the associated Dominating Set instance $G_{U,\mathcal{S}}$. As before, N_0 denotes the number of nodes of $G_{U,\mathcal{S}}$. The graph $\mathcal{G}_{\alpha,\beta}$ has a set of vertices $V_{U,\mathcal{S}} \cup X \cup W \cup \Gamma$. We have shown in Lemma 1 that the node degrees of vertices in $G_{U,\mathcal{S}}$ are within the interval $[N_0^a, N_0^b]$. The node degrees of the scaled instance $\mathcal{G}_{\alpha,\beta}^d$ are within $\left[N^{a/d}, N^{b/d}\right]$, where $N = N_0^d$ is the number of vertices of $G_{U,\mathcal{S}}^d$. We have to choose the parameters α, x, y and d of our construction in such a way that the following constraints are met.

(1) $\left|\left[N^{a/d}, N^{b/d}\right]\right| \geq N$.

(2) $|[x\Delta, y\Delta]| = o\left(N^{\frac{d-1}{d}}\right)$, where $N^{\frac{d-1}{d}}$ is a lower bound for the size of an optimum dominating set in $G_{U,\mathcal{S}}$.

(3) $\displaystyle\sum_{j=x\Delta}^{y\Delta} \left\lfloor \frac{e^\alpha}{j^\beta} \right\rfloor \cdot j = \mathrm{vol}(|x\Delta, y\Delta|) \geq |V_{\alpha,\beta}|$.

Constraint (1) ensures that the degree distribution of $G_{U,\mathcal{S}}^d$ fits into the power law distribution of $\mathcal{G}_{\alpha,\beta}$. Condition (2) enforces the set X to be *sufficiently small* compared to the size of a dominating set in $G_{U,\mathcal{S}}^d$. The last constraint (3) ensures that the total volume of the set $X = [x\Delta, y\Delta]$ is large enough such that X can dominate all vertices in $\mathcal{G}_{\alpha,\beta} \setminus G_{U,\mathcal{S}}^d$.

Note that constraint (1) is implied by the following stronger constraint: $e^\alpha \cdot N^{-b\beta/d} \geq N$. In all of the following cases, we work with this constraint instead of (1) and obtain the following bound for the parameter α: $e^\alpha \geq N^{1+b\beta/d}$. In order to minimize the value of the parameter α—and therefore the overall graph size—we choose $e^\alpha = N^{1+b\beta/d}$.

7.1 The Case $0 < \beta < 1$

Now we will give a detailed description of our lower bound construction for the case $0 < \beta < 1$. We start by providing estimates for the size and volumes of node intervals of the form $[x\Delta, y\Delta]$. Based on that we will then describe how to choose the parameters $0 < x < y \leq 1$ such as to satisfy the constraints (1)–(3) which we described in the previous section. Finally we will obtain the resulting approximation lower bound for the case $0 < \beta < 1$ in Theorem 1. The next lemma provides an estimate for the size of the interval $[x\Delta, y\Delta]$ and the volume $\mathrm{vol}([x\Delta, y\Delta])$.

Lemma 2. *Let $0 < \beta < 1$ and $X = [x\Delta, y\Delta]$. We have the following bounds on the size and the volume of the underlying interval:*

$$|[x\Delta, y\Delta]| \in \left[\frac{\Delta}{1-\beta}\left(y^{1-\beta} - x^{1-\beta}\right) - \left(\frac{1}{x^\beta} - \frac{1}{y^\beta}\right) - (y - x + 1),\ \frac{\Delta}{1-\beta}\left(y^{1-\beta} - x^{1-\beta}\right)\right]$$

$$\mathrm{vol}([x\Delta, y\Delta]) \geq \frac{\Delta^2}{2-\beta}\left(y^{2-\beta} - x^{2-\beta}\right) - \Delta\left(y^{1-\beta} - x^{1-\beta}\right) - \left(\frac{y\Delta(y\Delta-1)}{2} - \frac{x\Delta(x\Delta-1)}{2}\right)$$

Proof. We start by estimating the size of the interval $[x\Delta, y\Delta]$:

$$|[x\Delta, y\Delta]| = \sum_{j=x\Delta}^{y\Delta} \left\lfloor \frac{e^\alpha}{j^\beta} \right\rfloor \in \left[\sum_{j=x\Delta}^{y\Delta} \frac{e^\alpha}{j^\beta} - (y - x + 1)\Delta,\ \sum_{j=x\Delta}^{y\Delta} \frac{e^\alpha}{j^\beta}\right],$$

where

$$\sum_{j=x\Delta}^{y\Delta} \frac{e^\alpha}{j^\beta} \in \left[e^\alpha \int_{x\Delta}^{y\Delta} \frac{1}{j^\beta}\,\mathrm{d}j - e^\alpha \left(\frac{1}{(x\Delta)^\beta} - \frac{1}{(y\Delta)^\beta} \right),\ e^\alpha \int_{x\Delta}^{y\Delta} \frac{1}{j^\beta}\,\mathrm{d}j \right]$$

$$= \left[e^\alpha \left[\frac{j^{1-\beta}}{1-\beta} \right]_{x\Delta}^{y\Delta} - \frac{e^\alpha}{\Delta^\beta} \left(\frac{1}{x^\beta} - \frac{1}{y^\beta} \right),\ e^\alpha \left[\frac{j^{1-\beta}}{1-\beta} \right]_{x\Delta}^{y\Delta} \right]$$

$$= \left[\frac{e^\alpha \Delta^{1-\beta}}{1-\beta} \left(y^{1-\beta} - x^{1-\beta} \right) - \left(\frac{1}{x^\beta} - \frac{1}{y^\beta} \right),\ \frac{e^\alpha \Delta^{1-\beta}}{1-\beta} \left(y^{1-\beta} - x^{1-\beta} \right) \right]$$

$$= \left[\frac{\Delta}{1-\beta} \left(y^{1-\beta} - x^{1-\beta} \right) - \left(\frac{1}{x^\beta} - \frac{1}{y^\beta} \right),\ \frac{\Delta}{1-\beta} \left(y^{1-\beta} - x^{1-\beta} \right) \right].$$

In order to give a lower bound for the volume of the interval $[x\Delta, y\Delta]$, we have to take into account the rounding error resulting when we replace the sum $\sum_{x\Delta}^{y\Delta} \left\lfloor \frac{e^\alpha}{j^{\beta-1}} \right\rfloor$ by $\sum_{x\Delta}^{y\Delta} \frac{e^\alpha}{j^{\beta-1}}$. The sum of node degrees of nodes in $[x\Delta, y\Delta]$ is

$$\mathrm{vol}([x\Delta, y\Delta]) = \sum_{x\Delta}^{y\Delta} \left\lfloor \frac{e^\alpha}{j^\beta} \right\rfloor \cdot j \in \left[\sum_{x\Delta}^{y\Delta} \frac{e^\alpha}{j^{\beta-1}} - \underbrace{\left(\frac{y\Delta(y\Delta-1)}{2} - \frac{x\Delta(x\Delta-1)}{2} \right)}_{\text{rounding error}},\ \sum_{x\Delta}^{y\Delta} \frac{e^\alpha}{j^{\beta-1}} \right],$$

where

$$\sum_{x\Delta}^{y\Delta} \frac{e^\alpha}{j^{\beta-1}} \in \left[e^\alpha \int_{x\Delta}^{y\Delta} j^{1-\beta}\,\mathrm{d}j - e^\alpha \left((y\Delta)^{1-\beta} - (x\Delta)^{1-\beta} \right),\ e^\alpha \int_{x\Delta}^{y\Delta} j^{1-\beta}\,\mathrm{d}j \right]$$

$$= \left[e^\alpha \left[\frac{j^{2-\beta}}{2-\beta} \right]_{x\Delta}^{y\Delta} - \Delta \left(y^{1-\beta} - x^{1-\beta} \right),\ e^\alpha \left[\frac{j^{2-\beta}}{2-\beta} \right]_{x\Delta}^{y\Delta} \right]$$

$$= \left[\frac{\Delta^2}{2-\beta} \left(y^{2-\beta} - x^{2-\beta} \right) - \Delta \left(y^{1-\beta} - x^{1-\beta} \right),\ \frac{\Delta^2}{2-\beta} \left(y^{2-\beta} - x^{2-\beta} \right) \right].$$

This concludes the proof of the lemma. □

Now we choose $y = 1$ and obtain

$$|[x\Delta, \Delta]| \in \left[\frac{\Delta}{1-\beta} \left(1 - x^{1-\beta} \right) - \left(\frac{1}{x^\beta} - 1 \right) - (2-x)\Delta,\ \frac{\Delta}{1-\beta} \left(1 - x^{1-\beta} \right) \right].$$

The volume of that interval is then estimated as

$$\begin{aligned}\operatorname{vol}([x\Delta,\Delta]) &\geq \frac{\Delta^2}{2-\beta}\left(1-x^{2-\beta}\right)-\Delta\left(1-x^{1-\beta}\right)-\left(\frac{\Delta(\Delta+1)}{2}-\frac{x^2\Delta^2-x\Delta}{2}\right)\\ &= \frac{\Delta^2}{2-\beta}\left(1-x^{2-\beta}\right)-\frac{\Delta^2}{2}+\frac{x^2}{2}\Delta^2-\Delta\left(1-x^{1-\beta}-\frac{1}{2}+\frac{x}{2}\right)\\ &= \Delta^2\left(\frac{1-x^{2-\beta}}{2-\beta}-\frac{1}{2}+\frac{x^2}{2}\right)-\Delta\left(1-x^{1-\beta}-\frac{1}{2}+\frac{x}{2}\right).\end{aligned}$$

Now we use the scaling technique with scaling parameter d. Thus we have to choose α such that $e^\alpha \geq N^{\frac{d+b\beta}{d}}$. Since $N^{\frac{d-1}{d}}$ is a lower bound for the optimum in $G^d_{U,\mathcal{S}}$, we have $N^{\frac{d-1}{d}} = e^{\frac{d-1}{d+b\beta}\cdot\alpha} = e^{(1-\delta)\alpha}$, where we can choose $1-\delta$ arbitrary close to 1 by choosing d sufficiently large. The size of the interval $[x\Delta,\Delta]$ is of order $\Delta(1-x^{1-\beta})$, hence we want to choose x such that $\Delta(1-x^{1-\beta}) = e^{\alpha/\beta}\cdot e^p$ with $\alpha/\beta \cdot p < (1-\delta)\alpha$, i.e. $p < (1-\delta)\beta$. So suppose we choose x such that $p = (1-\delta')\beta$, where $1-\delta'$ can be chosen arbitrary close to 1. Furthermore, the interval $[x\Delta,\Delta]$ needs to provide sufficient volume to dominate the rest of the graph, i.e. (using Lemma 2) we require that $\Delta^2\left(\frac{1}{2-\beta}-\frac{1}{2}-x^{2-\beta}\left(\frac{1}{2-\beta}-\frac{x^\beta}{2}\right)\right) > \Delta$. This yields the requirement $\frac{1}{2-\beta}-\frac{1}{2}-x^{2-\beta}\left(\frac{1}{2-\beta}-\frac{x^\beta}{2}\right) > \frac{1}{\Delta}$, which is implied by $1-\frac{1}{\Delta\left(\frac{1}{2-\beta}-\frac{1}{2}\right)} > x^2$. Combining this with the upper bound requirement for the size of the interval, we obtain

$$\left(1-\frac{1-\beta}{e^{\alpha\left(\frac{1}{\beta}-(1-\delta')\right)}}\right)^{\frac{1}{1-\beta}} \leq x < \left(1-\frac{1}{\left(\frac{1}{2-\beta}-\frac{1}{2}\right)\cdot e^{\frac{\alpha}{\beta}}}\right)^{\frac{1}{2}}. \tag{7.3}$$

We observe that $\frac{1}{1-\beta} > 1 > \frac{1}{2}$ for $\beta \in (0,1)$, and furthermore $\frac{\alpha}{\beta}-(1-\delta')\alpha < \frac{\alpha}{\beta}$. Hence we can choose x such that Equation 7.3 holds. Thus for this choice of x we have $|[x\Delta,\Delta]| = o\left(N^{\frac{d-1}{d}}\right)$ and $\operatorname{vol}([x\Delta,\Delta]) \geq |\mathcal{G}_{\alpha,\beta}|$, fulfilling the constraints (2) and (3) in the graph $\mathcal{G}_{\alpha,\beta}$.

Resulting Lower Bound. In the case $0 < \beta < 1$, we have $|\mathcal{G}_{\alpha,\beta}| = \frac{e^{\alpha/\beta}}{1-\beta}$ and we choose α such that $e^\alpha = N^{1+\frac{b\beta}{d}}$. We have $\mathsf{OPT}(\mathcal{G}_{\alpha,\beta}) = (1+o(1))\mathsf{OPT}(G^d_{U,\mathcal{S}})$, and furthermore $N = (|\mathcal{G}_{\alpha,\beta}|\cdot(1-\beta))^{\frac{d\beta}{d+b\beta}}$. Now using the inequalities 7.1 and 7.2, we obtain the following bounds on the size of an optimum dominating set for $\mathcal{G}_{\alpha,\beta}$: If $|\mathcal{G}_{\alpha,\beta}|\cdot(1-\beta) = \phi$, then

$$\begin{aligned}&|\mathsf{OPT}(\mathcal{G}_{\alpha,\beta})| \leq k\left(\phi^{\frac{\beta(d-1+\frac{\varepsilon}{2+\varepsilon})}{d+b\beta}}\right) \quad \text{or}\\ &|\mathsf{OPT}(\mathcal{G}_{\alpha,\beta})| \geq k\left(\phi^{\frac{\beta(d-1+\frac{\varepsilon}{2+\varepsilon})}{d+b\beta}}\right)\frac{(1-\varepsilon)\varepsilon}{2+\varepsilon}\left(\frac{1}{2}\right)^{\frac{\varepsilon}{2+\varepsilon}}\left(\ln\left(\phi^{\frac{\beta}{d+b\beta}}\right)-O(1)\right).\end{aligned}$$

Altogether we obtain the following result.

Theorem 1. *For $0 < \beta < 1$, the* Min-DS *problem on (α, β)-PLGs is hard to approximate within*

$$\frac{(1-\varepsilon)\varepsilon}{2+\varepsilon} \cdot \left(\frac{1}{2}\right)^{\frac{\varepsilon}{2+\varepsilon}} \cdot \left(\frac{\beta}{d+b\beta} \cdot \left(\ln(|\mathcal{G}_{\alpha,\beta}|) - \ln\left(\frac{1}{1-\beta}\right)\right) - O(1)\right) \ .$$

7.2 The Case $\beta = 1$

In the case $\beta = 1$, we can omit the scaling and directly embed the graph $G_{U,\mathcal{S}}$ into a PLG $\mathcal{G}_{\alpha,\beta}$. Formally this means that we choose $d = 1$ in the scaling construction and we have the condition $|[x\Delta, y\Delta]| = o\left(N_0^{\frac{\varepsilon}{2+\varepsilon}}\right)$ in constraint (2). It suffices to describe the choice of parameters x and α for a given $G_{U,\mathcal{S}}$ and to verify that the constraints (1)–(3) of the graph $\mathcal{G}_{\alpha,\beta}$ are satisfied. It turns out that if we choose x such that $\ln(1/x) = o(e^{\alpha \cdot \frac{b}{1+b}})$ and $N_0^{1+b} = e^{\alpha}$, we obtain the following lower bound.

Theorem 2. *For $\beta = 1$, the* Min-DS *problem on (α, β)-PLGs is hard to approximate within*

$$\frac{(1-\varepsilon)\varepsilon}{2+\varepsilon} \cdot \left(\frac{1}{2}\right)^{\frac{\varepsilon}{2+\varepsilon}} \cdot \left(\frac{(1-o(1))\ln(|\mathcal{G}_{\alpha,\beta}|)}{1+b} - O(1)\right) \ .$$

In order to prove the theorem, we have to describe the choice of parameters x and α for a given $G_{U,\mathcal{S}}$ such as to satisfy the constraints (1)–(3). First we will provide estimates for the size and volume of node intervals $X = [x\Delta, y\Delta]$ in the following lemma. We will restrict ourselves to the case $y = 1$.

Lemma 3. *Let $\beta = 1$ and $X = [x\Delta, \Delta]$. We have the following bounds on the size and the volume of the interval:*

$$|[x\Delta, \Delta]| \in \left[e^{\alpha} \ln\left(\frac{1}{x}\right) - \left(\frac{1}{x} - 1\right),\ e^{\alpha} \ln\left(\frac{1}{x}\right)\right] \quad \textit{and}$$

$$\mathrm{vol}([x\Delta, \Delta]) \in \left[\Delta^2 \left(\frac{1}{2} - x + \frac{x^2}{2}\right) - \frac{1-x}{2}\Delta,\ (1-x)\Delta^2\right].$$

Proof. For a given $x \in [0,1]$, the size of the interval $[x\Delta, \Delta] = \{\, v \in V(\mathcal{G}_{\alpha,\beta}) \mid x\Delta \le \mathsf{deg}_{\alpha,\beta}(v) \le \Delta \,\}$ satisfies

$$\begin{aligned} |[x\Delta, \Delta]| &\in \left[\sum_{xe^{\alpha}}^{e^{\alpha}} \frac{e^{\alpha}}{j} - (1-x)e^{\alpha},\ \sum_{xe^{\alpha}}^{e^{\alpha}} \frac{e^{\alpha}}{j}\right] \\ &\subseteq \left[e^{\alpha}\left(\ln(e^{\alpha}) - \ln(xe^{\alpha})\right) - e^{\alpha}\left(\frac{1}{x} - 1\right) \cdot \frac{1}{e^{\alpha}},\ e^{\alpha} \cdot \ln\left(\frac{1}{x}\right)\right] \\ &= \left[e^{\alpha} \ln\left(\frac{1}{x}\right) - \left(\frac{1}{x} - 1\right),\ e^{\alpha} \ln\left(\frac{1}{x}\right)\right]. \end{aligned}$$

The volume of that interval is

$$\begin{aligned}
\operatorname{vol}([x\Delta,\Delta]) &\in \left[\sum_{x\Delta}^{\Delta} e^{\alpha} - j,\ \sum_{x\Delta}^{\Delta} e^{\alpha}\right] \\
&\subseteq \left[e^{\alpha}(1-x)\Delta - \left(\frac{\Delta(\Delta+1)}{2} - \frac{x\Delta(x\Delta+1)}{2}\right),\ e^{\alpha}(1-x)\Delta\right] \\
&= \left[\Delta^2\left(\frac{1}{2} - x + \frac{x^2}{2}\right) - \frac{1-x}{2}\Delta,\ (1-x)\Delta^2\right].
\end{aligned}$$

□

Proof of Theorem 2. From Lemma 3, we obtain that for every $x < 1$ being bounded away from 1, the volume of the interval $[x\Delta, \Delta]$ is $\omega(|G_{\alpha,1}|)$. Recall that in order to achieve $N_0 \le \left|\left[N_0^a, N_0^b\right]\right|$, it suffices to choose α sufficiently large such that $N_0 \le \frac{e^{\alpha}}{N_0^{b\beta}} = \frac{e^{\alpha}}{N_0^{b}}$. Hence suppose we have $N_0^{1+b} = e^{\alpha}$. This implies $\frac{e^{\alpha}}{N_0^b} = e^{\alpha\cdot\frac{1}{1+b}}$. Thus it suffices to choose x such that $\ln\left(\frac{1}{x}\right) = o\left(e^{\alpha\cdot\frac{b}{1+b}}\right)$. The size of the PLG is $|\mathcal{G}_{\alpha,\beta}| = \alpha e^{\alpha}$, and from $N_0^{1+b} = e^{\alpha}$ we obtain $N_0 = e^{\frac{\alpha}{1+b}} = \left(\frac{|\mathcal{G}_{\alpha,\beta}|}{\ln(|\mathcal{G}_{\alpha,\beta}|)}\right)^{\frac{1}{1+b}}$. Hence, we obtain the same lower bound as for the case $0 < \beta < 1$ stated in Theorem 1. This concludes the proof of Theorem 2. □

7.3 The Case $1 < \beta \le 2$

In this section we consider the case $1 < \beta \le 2$. Again we start with giving sufficiently strong estimates of interval sizes and volumes for node intervals of the form $[x\Delta, y\Delta]$. Then we will show how to choose the parameters x and y such that the reduction from the Set Cover Problem works. We start with the sub-case $1 < \beta < 2$. The following lemma provides estimates for the sizes and volumes of node intervals of the form $[x\Delta, y\Delta]$ for the special case $y = 1$.

Lemma 4. *Let $1 < \beta < 2$ and $X = [x\Delta, \Delta]$. We have the following bounds on the size and the volume of the interval:*

$$|[x\Delta,\Delta]| \le \Delta\frac{1-x}{x^{\beta}}$$

and

$$\operatorname{vol}([x\Delta,\Delta]) \ge (1-o(1))\Delta^2 \cdot \frac{\beta - 2x^{2-\beta} + (2-\beta)x^2}{2\cdot(2-\beta)} \ .$$

Proof. For $1 < \beta < 2$, we have the following estimate for the size of a node interval $[x\Delta, y\Delta]$:

$$|[x\Delta, y\Delta]| = \sum_{j=y\Delta}^{x\Delta} \left\lfloor \frac{e^\alpha}{j^\beta} \right\rfloor \in \left[\frac{e^\alpha}{\Delta^\beta}(y-x)\Delta \frac{1}{y^\beta} - (y-x)\Delta,\ \frac{e^\alpha}{\Delta^\beta}(y-x)\Delta \frac{1}{x^\beta} \right]$$
$$= \left[\Delta(y-x)\left(\frac{1}{y^\beta} - 1\right),\ \Delta \frac{y-x}{x^\beta} \right].$$

The volume $\mathrm{vol}(|x\Delta, y\Delta|) = \sum_{j=x\Delta}^{y\Delta} \left\lfloor \frac{e^\alpha}{j^\beta} \right\rfloor \cdot j$ can be estimated as follows:

$$\begin{aligned}
\mathrm{vol}(|x\Delta, y\Delta|) &\geq e^\alpha \sum_{j=x\Delta}^{y\Delta} j^{1-\beta} - r_\beta \\
&= (1-o(1))e^\alpha \cdot \int_{x\Delta}^{y\Delta} j^{1-\beta}\,\mathrm{d}j - r_\beta \\
&= (1-o(1))e^\alpha \cdot \left[\frac{j^{2-\beta}}{2-\beta}\right]_{x\Delta}^{y\Delta} - r_\beta \\
&= (1-o(1))e^\alpha \cdot e^{\alpha\frac{2-\beta}{\beta}} \cdot \frac{y^{2-\beta} - x^{2-\beta}}{2-\beta} - r_\beta \\
&= (1-o(1))\Delta^2 \cdot \frac{y^{2-\beta} - x^{2-\beta}}{2-\beta} - r_\beta,
\end{aligned}$$

where $r_\beta = \frac{\Delta^2(y^2-x^2)}{2} + \frac{\Delta(y+x)}{2}$ is an upper bound for the rounding error. We conclude that $\mathrm{vol}([x\Delta, y\Delta]) = \omega(|\mathcal{G}_{\alpha,\beta}|)$, provided we choose x and y in such a way that $\frac{y^{2-\beta}-x^{2-\beta}}{2-\beta} - r_\beta > 0$. Let us choose $y = 1$. Then, we have

$$\frac{y^{2-\beta} - x^{2-\beta}}{2-\beta} - r_\beta = \frac{1 - x^{2-\beta}}{2-\beta} - \frac{1-x^2}{2} - o(1) = \frac{\beta - 2x^{2-\beta} + (2-\beta)x^2}{2\cdot(2-\beta)} - o(1).$$

$\square$

Now we want to make use of the estimates provided in the previous lemma. We have to choose the parameter x of our construction in such a way that the constraints (1)–(3) are satisfied. In order to satisfy (3), it suffices to choose $x \in (0,1)$ such that $\beta - 2x^{2-\beta} + (2-\beta)x^2 > 0$. This inequality holds for $x < (\beta/2)^{\frac{1}{2-\beta}}$, since $\frac{\beta}{2} < 1$. For our choice of α, we have that $N^{\frac{d-1}{d}} = e^{\alpha\cdot\frac{d-1}{d+b\beta}}$, and hence constraint (2) holds if the following constraint is satisfied: $\Delta \cdot \frac{y-x}{x^\beta} = \frac{y-x}{x^\beta} \cdot e^{\frac{\alpha}{\beta}} = o(e^{\alpha\cdot\frac{d-1}{d+b\beta}})$. Hence, for our choice of $y = 1$ and $x < (\beta/2)^{\frac{1}{2-\beta}}$, this last constraint is satisfied if $\frac{\alpha}{\beta} < \alpha \cdot \frac{d-1}{d+b\beta}$, i. e. when $d > \frac{(b+1)\beta}{\beta-1}$.

We proceed similarly in the case $\beta = 2$ and obtain the following version of Lemma 4.

Lemma 5. *Let $\beta = 2$ and $X = [x\Delta, y\Delta]$. We have the following bounds on the size and the volume* *of the interval:*

$$|[x\Delta, y\Delta]| \in \left[\sqrt{e^{\alpha}} \cdot \frac{y-x}{y^{\beta}}, \sqrt{e^{\alpha}} \cdot \frac{y-x}{x^{\beta}}\right] \quad \textit{and} \quad \mathrm{vol}([x\Delta, y\Delta]) = (1-o(1))e^{\alpha}\left(\ln\left(\frac{1}{x}\right) - \ln\left(\frac{1}{y}\right)\right).$$

Proof. We give an estimate of the size of the interval $[x\Delta, y\Delta]$ and of the volume of that interval. We have that

$$|[x\Delta, y\Delta]| \in \left[\Delta\frac{y-x}{y^{\beta}}, \Delta\frac{y-x}{x^{\beta}}\right] = \left[\sqrt{e^{\alpha}} \cdot \frac{y-x}{y^{\beta}}, \sqrt{e^{\alpha}} \cdot \frac{y-x}{x^{\beta}}\right].$$

The value $\mathrm{vol}([x\Delta, y\Delta])$ of the interval is $(1-o(1))\sum_{j=x\Delta}^{y\Delta} \frac{e^{\alpha}}{j^{\beta}} j = (1-o(1))e^{\alpha}\left(\ln(y\Delta) - \ln(x\Delta)\right)$ $= (1-o(1))e^{\alpha}\left(\ln\left(\frac{1}{x}\right) - \ln\left(\frac{1}{y}\right)\right)$. □

Now we use these estimates in order to choose parameters x and y for the case $\beta = 2$. First we choose $y = 1$. Thus we obtain

$$\mathrm{vol}([x\Delta, y\Delta]) = (1-o(1))\sum_{j=x\Delta}^{y\Delta} \frac{e^{\alpha}}{j^{\beta}} \cdot j = (1-o(1))e^{\alpha}\left(\ln\left(\frac{1}{x}\right) - 0\right).$$

Hence, we choose x such that $\ln\left(\frac{1}{x}\right) \geq \zeta(\beta)$, i.e. $x \leq \frac{1}{e^{\zeta(\beta)}}$. Then the volume of the interval $[x\Delta, \Delta]$ suffices to dominate the rest of the graph and constraint (3) is satisfied. The size of the interval $[x\Delta, \Delta]$ satisfies $|[x\Delta, \Delta]| \in \left[\Delta\frac{1-x}{1}, \Delta\frac{1-x}{x^{\beta}}\right]$. The two intervals $[x\Delta, \Delta]$ and $[N^{a/d}, N^{b/d}]$ need to be node disjoint. Hence, we want to choose d such that $N^{b/d} < x\Delta$. For $x = \frac{1}{e^{\zeta(\beta)}}$, we have $x\Delta = e^{\alpha/\beta - \zeta(\beta)}$. Furthermore, the size N of the graph $G_{U,\mathcal{S}}^{d}$ satisfies $N = |G_{U,\mathcal{S}}^{d}| \leq e^{\alpha\frac{d}{d+b\beta}}$. This yields the following bound for the scaling parameter d: $N^{b/d} < x\Delta \iff e^{\alpha b \cdot \frac{1}{d+b\beta}} < e^{\alpha/\beta - \zeta(\beta)} \iff d > \frac{\alpha \cdot b}{\alpha/\beta - \zeta(\beta)} - b\beta$.

Resulting Lower Bound. Since $|\mathcal{G}_{\alpha,\beta}| = \zeta(\beta) \cdot e^{\alpha}$ and the parameter α is chosen such that $e^{\alpha} = N^{1+\frac{b\beta}{d}}$, we have $|\mathcal{G}_{\alpha,\beta}| = \zeta(\beta) \cdot N^{1+\frac{b\beta}{d}}$. This yields $N = \left(\frac{|\mathcal{G}_{\alpha,\beta}|}{\zeta(\beta)}\right)^{\frac{d}{d+b\beta}}$. Now using inequalities 7.1 and 7.2 we obtain the following bounds on the size of an optimum dominating set for $\mathcal{G}_{\alpha,\beta}$: If $\frac{|\mathcal{G}_{\alpha,\beta}|}{\zeta(\beta)} = \phi$, then

$$|\mathsf{OPT}(\mathcal{G}_{\alpha,\beta})| \leq \left(\phi^{\frac{d}{d+b\beta}}\right)^{\frac{d-1}{d}} k\left(\phi^{\frac{d}{d+b\beta}}\right)^{\frac{1}{d}\frac{\varepsilon}{2+\varepsilon}} = k\left(\phi^{\frac{d-1+\frac{\varepsilon}{2+\varepsilon}}{d+b\beta}}\right) \quad \text{or}$$

$$|\mathsf{OPT}(\mathcal{G}_{\alpha,\beta})| \geq k\left(\phi^{\frac{d-1+\frac{\varepsilon}{2+\varepsilon}}{d+b\beta}}\right)\frac{(1-\varepsilon)\varepsilon}{2+\varepsilon}\left(\frac{1}{2}\right)^{\frac{\varepsilon}{2+\varepsilon}}\left(\ln\left(\phi^{\frac{d}{d+b\beta}\frac{1}{d}}\right) - O(1)\right).$$

Altogether, we obtain the following theorem.

Theorem 3. *For $1 < \beta \leq 2$, the* MIN-DS *problem on (α, β)-power law graphs is hard to approximate within*

$$\frac{(1-\varepsilon)\cdot\varepsilon}{2+\varepsilon}\cdot\left(\frac{1}{2}\right)^{\frac{\varepsilon}{2+\varepsilon}}\cdot\left(\frac{\ln\left(|\mathcal{G}_{\alpha,\beta}|\right)-\ln(\zeta(\beta))}{d+b\beta}-O(1)\right) .$$

8 New Upper Bounds for $\beta > 2$

Now we are going to prove new upper bounds for MIN-DS for $\beta > 2$. It was already observed by Shen et al. [21] that, in the case of $\beta > 2$, the MIN-DS problem on (α, β)-PLG is in the class APX. They showed that there exists an efficient approximation algorithm with approximation ratio $(\zeta(\beta) - 1/2)/(\zeta(\beta) - \sum_{j=1}^{t_0} 1/j^\beta)$ for some $t_0 = O(1)$. In this section we give a new and explicit upper bound, based on our techniques of estimating sizes and volumes of intervals in (α, β)-PLG. Let us first give an outline of our underlying ideas. The approximation ratio $(\zeta(\beta) - 1/2)/(\zeta(\beta) - \sum_{j=1}^{t_0} 1/j^\beta)$ from [21] comes from the observation that in the construction of a dominating set, we can omit at least half of the degree 1 nodes. The worst case occurs when the degree 1 nodes are connected by a matching in $\mathcal{G}_{\alpha,\beta}$. In the denominator, t_0 is chosen as small as possible such that the volume of nodes of degree at least $t_0 + 1$ in $\mathcal{G}_{\alpha,\beta}$ suffices to dominate all the nodes of degree at most t_0.

We will now give an improved approximation ratio, based on a more careful analysis (cf. [10]). We will also construct upper and lower bounds for the size of a dominating set in $\mathcal{G}_{\alpha,\beta}$, but we will relate these bounds to each other. The lower bound on the size of a dominating set in $\mathcal{G}_{\alpha,\beta}$ given in part (ii) of the following lemma was also used in [21].

Lemma 6.

(i) If $\mathrm{vol}([x\Delta, \Delta]) = \sum_{j=x\Delta}^{\Delta} \left\lfloor \frac{e^\alpha}{j^\beta} \right\rfloor \cdot j < \lfloor e^\alpha \rfloor$, *then* $|[x\Delta, \Delta]|$ *is a lower bound on the size of a dominating set in* $\mathcal{G}_{\alpha,\beta}$.

(ii) If $\mathrm{vol}([x\Delta, \Delta]) = \sum_{j=x\Delta}^{\Delta} \left\lfloor \frac{e^\alpha}{j^\beta} \right\rfloor \cdot j < \sum_{j=1}^{x\Delta-1} \left\lfloor \frac{e^\alpha}{j^\beta} \right\rfloor$, *then* $|[x\Delta, \Delta]|$ *is a lower bound on the size of a dominating set in* $\mathcal{G}_{\alpha,\beta}$.

Proof. Considering (i), let D be a dominating set in $\mathcal{G}_{\alpha,\beta}$, and let $D_1 = D \cap [x\Delta, \Delta]$ and $D_2 = D \setminus D_1$. Suppose $|D_2| < |[x\Delta, \Delta] \setminus D_1|$. Since $\forall v \in D_2, u \in [x\Delta, \Delta] \setminus D_1$ we have $\mathsf{deg}_{\alpha,\beta}(v) < \mathsf{deg}_{\alpha,\beta}(u)$, this implies $\mathrm{vol}(D_2) < \mathrm{vol}([x\Delta, \Delta] \setminus D_1)$ and thus $\mathrm{vol}(D) < \mathrm{vol}([x\Delta, \Delta]) < \lfloor e^\alpha \rfloor$, a contradiction.

Suppose in case (ii) that $\mathrm{vol}([x\Delta, \Delta]) < |[1, x\Delta - 1]|$ and that D, D_1, D_2 are the same as in the proof of (i). Again we obtain $\mathrm{vol}(D_2) < \mathrm{vol}([x\Delta, \Delta] \setminus D_1)$, which implies $\mathrm{vol}(D) < \mathrm{vol}([x\Delta, \Delta]) < |[1, x\Delta - 1]$. Thus the volume of D is not sufficient to dominate the subset $[1, x\Delta - 1]$, a contradiction. □

We will now analyze upper bounds for the approximability of MIN-DS based on the lower bounds from Lemma 6. Instead of just giving upper and lower bounds on the size of an optimum dominating set and a greedy solution separately, we will explicitly relate upper and lower bound to each other. Let $\mathcal{G}_{\alpha,\beta}$ be an (α, β)-PLG with $\beta > 2$. Let W be the set of neighbors of degree 1 nodes of degree at least 2 in $\mathcal{G}_{\alpha,\beta}$ and let M be the set of degree 1 nodes in $\mathcal{G}_{\alpha,\beta}$ which are adjacent to another degree 1 node. Let $R = V \setminus (W \cup \{v \in V \mid \mathsf{deg}_{\alpha,\beta}(v) = 1\})$. Then there exists some $c = c_\beta > 0$ not depending on α such that $|W| \geq c \cdot e^\alpha$. This implies $|R| \leq (\zeta(\beta) - c - 1)e^\alpha$.

Lemma 7. *If $\mathcal{G}_{\alpha,\beta}$ is a connected (α, β)-PLG with $\beta > 2$ and W and R are defined as above, then there exists an optimum dominating set OPT in $\mathcal{G}_{\alpha,\beta}$ with $\mathsf{OPT} = \mathsf{OPT}_R \cup W \cup M'$, where OPT_R is an optimum dominating set for the induced subgraph $\mathcal{G}_{\alpha,\beta}[R]$ on R and $M' \subset M$ is of cardinality $|M'| = \frac{|M|}{2}$.*

The maximum degree in the subgraph $\mathcal{G}_{\alpha,\beta}[R]$ induced by R is at most Δ. We consider the dominating set $D = W \cup D_{Gr} \cup M'$ where D_{Gr} is a dominating set for $\mathcal{G}_{\alpha,\beta}[R]$ constructed by the greedy algorithm and $M' \subset M$ is a subset of size $\frac{|M|}{2}$ dominating M. Since $R = V \setminus (W \cup V_1)$ and $|\mathsf{OPT}_R| \leq |R|$, the approximation ratio is at most

$$\max \left\{ \frac{r \cdot |\mathsf{OPT}_R| + |W| + \frac{|M|}{2}}{|\mathsf{OPT}_R| + |W| + \frac{|M|}{2}} \;\middle|\; \begin{array}{l} |\mathsf{OPT}_R| \leq |R|, \\ r = \min\left\{\frac{\alpha}{\beta}, \frac{|R|}{|\mathsf{OPT}_R|}\right\} \end{array} \right\} .$$

Case 1: $\left(r = \frac{\alpha}{\beta}\right)$ This means that $\frac{\alpha}{\beta} \leq \frac{|R|}{|\mathsf{OPT}_R|}$, i.e. $|\mathsf{OPT}_R| \leq \frac{\beta}{\alpha} \cdot |R|$. The upper bound for the approximation ratio is monotone increasing in $|\mathsf{OPT}_R|$, hence it is bounded by

$$\frac{\frac{\alpha}{\beta} \cdot \frac{\beta}{\alpha} \cdot |R| + |W| + \frac{|M|}{2}}{\frac{\beta}{\alpha} \cdot |R| + |W| + \frac{|M|}{2}} = \frac{|R| + |W| + \frac{|M|}{2}}{\frac{\beta}{\alpha} \cdot |R| + |W| + \frac{|M|}{2}} .$$

Case 2: $\left(r = \frac{|R|}{|OPT_R|} < \frac{\alpha}{\beta}\right)$ Here, we have $|\mathsf{OPT}_R| > \frac{\beta \cdot |R|}{\alpha}$ and obtain

$$\frac{r \cdot |\mathsf{OPT}_R| + |W| + \frac{|M|}{2}}{|\mathsf{OPT}_R| + |W| + \frac{|M|}{2}} = \frac{|R| + |W| + \frac{|M|}{2}}{|\mathsf{OPT}_R| + |W| + \frac{|M|}{2}} \leq \frac{|R| + |W| + \frac{|M|}{2}}{\frac{\beta}{\alpha} \cdot |R| + |W| + \frac{|M|}{2}} .$$

Thus we have shown that the approximation ratio is bounded by $\frac{|R|+|W|+\frac{|M|}{2}}{\frac{\beta}{\alpha}\cdot|R|+|W|+\frac{|M|}{2}}$. Now we need to construct an upper bound for this term. We consider two cases.

Case I: $(\zeta(\beta - 1) - 1 < 1)$ In this case, the volume of nodes of degree at least 2 does not suffice to dominate all the degree 1 nodes. Hence in this case, $M \neq \emptyset$. We obtain the following lower bound for the cardinality of M: $|M| \geq e^\alpha - (\zeta(\beta - 1) - 1)e^\alpha = (2 - \zeta(\beta - 1))e^\alpha$. Nevertheless we will use the upper bound $|R| \leq (\zeta(\beta) - 1)e^\alpha$. Since the term $\frac{|R|+|W|+\frac{|M|}{2}}{\frac{\beta}{\alpha}\cdot|R|+|W|+\frac{|M|}{2}}$ is monotone increasing in $|R|$, we obtain

$$\rho(\beta) = \frac{|R| + |W| + \frac{|M|}{2}}{\frac{\beta}{\alpha} \cdot |R| + |W| + \frac{|M|}{2}} \leq \frac{(\zeta(\beta)-1)e^\alpha + \frac{(2-\zeta(\beta-1))e^\alpha}{2}}{\frac{\beta}{\alpha}(\zeta(\beta)-1)e^\alpha + \frac{(2-\zeta(\beta-1))e^\alpha}{2}} = \frac{\zeta(\beta) - \frac{\zeta(\beta-1)}{2}}{1 - \frac{\zeta(\beta-1)}{2}} = \frac{\zeta(\beta-1) - 2\zeta(\beta)}{\zeta(\beta-1) - 2}.$$

Case II: $(\zeta(\beta-1) - 1 \geq 1)$ In this case, the volume of the nodes of degree at least 2 suffices to dominate the degree 1 nodes. Now, we construct a lower bound for $|W|$ as follows:

$$\begin{aligned} |W| &\geq \min\{\, |[d,\Delta]| \mid \mathrm{vol}([d,\Delta]) > e^\alpha \,\} \\ &= \min\left\{ \left(\zeta(\beta) - \sum_{j=1}^{d-1} \frac{1}{j^\beta}\right) e^\alpha \;\middle|\; \left(\zeta(\beta-1) - \sum_{j=1}^{d-1} \frac{1}{j^{\beta-1}}\right) e^\alpha > e^\alpha \right\} . \end{aligned}$$

Hence, in this case, the approximation ratio is bounded by

$$\rho'(\beta) = \frac{\zeta(\beta)-1}{\frac{\beta}{\alpha} \cdot |[1,d-1]| + |[d,\Delta]|} = \frac{\zeta(\beta)-1}{\zeta(\beta) - \sum_{j=1}^{d-1} \frac{1}{j^\beta}} ,$$

where $d = \min\{\, d' \mid \mathrm{vol}([d',\Delta]) > e^\alpha \,\}$.

Altogether, we obtain the following theorem.

Theorem 4. *For $2 < \beta \leq 2.729$, the* Min-DS *problem on (α,β)-power law graph is approximable within approximation ratio $\rho'(\beta)$ and for $\beta > 2.729$ within approximation ratio $\rho(\beta)$, where $d = \min\{\, d' \mid \mathrm{vol}([d',\Delta]) > e^\alpha \,\}$ and*

$$\rho'(\beta) = \frac{\zeta(\beta)-1}{\zeta(\beta) - \sum_{j=1}^{d-1} \frac{1}{j^\beta}} \quad \textit{and} \quad \rho(\beta) = \frac{\zeta(\beta) - \frac{\zeta(\beta-1)}{2}}{1 - \frac{\zeta(\beta-1)}{2}} .$$

In Figure 6 we present a plot of the above approximation ratios $\rho(\beta)$ and $\rho'(\beta)$ in the valid ranges for certain choices of the parameter d.
In what follows, we are going to analyze the functional dependencies of a parameter β.

9 The Functional Case $\beta_f = 2 + f(n)^{-1}$

So far we know that for every constant $\beta \leq 2$, there is a logarithmic lower bound for the approximability of MIN-DS on (α,β)-PLGs, while for every $\beta > 2$ the problem is constant factor approximable. In this section we will take a closer look at this phase transition at the point $\beta = 2$. We consider the case when the parameter β is a function $\beta = 2 + 1/f(n)$ of the size n of the power law graph, converging to 2 from above. This can be seen as a combinatorial variant of preferential attachment PLG (cf. also [11]). Surprisingly we will obtain a very tight phase transition of the

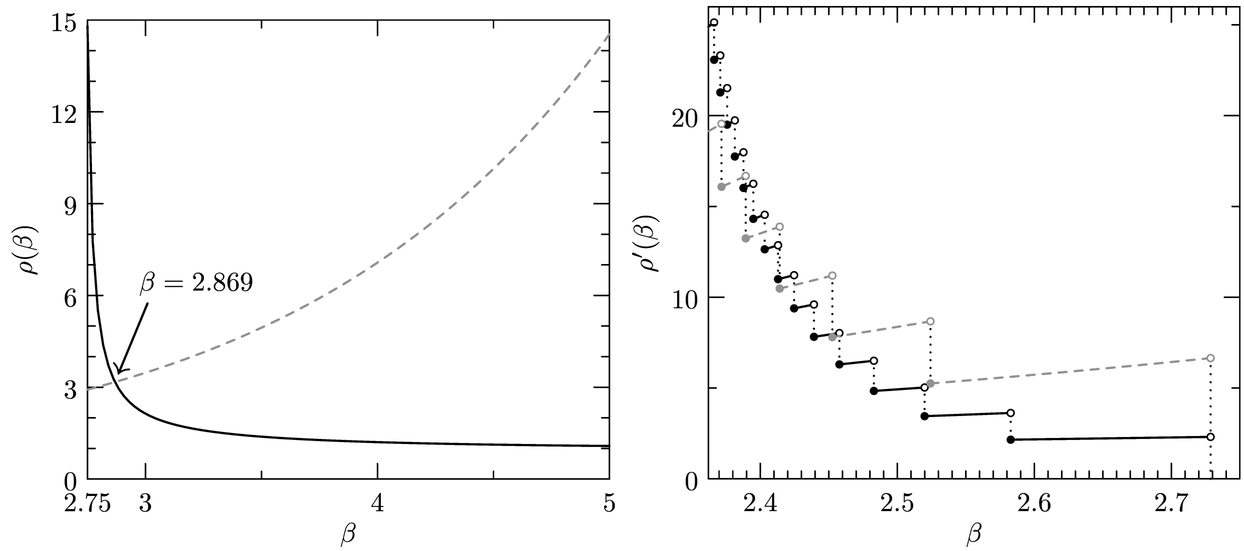

Figure 6: Plot of the approximation ratios $\rho(\beta)$ and $\rho'(\beta)$ (solid line) in the valid ranges for certain choices of the parameter $d = \min\{\, d' \mid \mathrm{vol}([d', \Delta]) > e^{\alpha} \,\}$, in comparison to the results of Shen et al. [21] (dashed line)

computational complexity of the problem, depending on the convergence rate of the function. Let us first give a precise description of the model.

Let $f \colon \mathbb{N} \to \mathbb{N}$ be a monotone increasing unbounded function. For $\beta_f = 2 + f(n)^{-1}$, an (α, β_f)-PLG is an undirected multigraph $\mathcal{G}_{\alpha,\beta_f}$ with n nodes and maximum degree $\Delta_f = \left\lfloor e^{\alpha/\beta_f} \right\rfloor$ such that for $j = 1, \ldots, \Delta_f = \left\lfloor e^{\alpha/\beta_f} \right\rfloor$, the number of nodes of degree j in $\mathcal{G}_{\alpha,\beta_f}$ equals $\left\lfloor \frac{e^{\alpha}}{j^{\beta_f}} \right\rfloor$. Especially this means that $\sum_{j=1}^{\Delta_f} \left\lfloor \frac{e^{\alpha}}{j^{2+1/f(n)}} \right\rfloor = n$.

We consider two cases for $\beta_f = 2 + f(n)^{-1}$, namely, $f(n) = \omega(\log(n)$ and $f(n) = o(\log(n))$. Let us first consider the case $f(n) = \omega(\log(n))$. In this case, β_f is a function of the size n of the graph which converges to 2 sufficiently fast. We obtain the following theorem.

Theorem 5. *For $\beta_f = 2 + f(n)^{-1}$ with $f(n) = \omega(\log(n))$, the* Min-DS *problem on (α, β_f)-PLG is hard to approximate within*

$$\frac{(1-\varepsilon)\cdot\varepsilon}{2+\varepsilon} \cdot \left(\frac{1}{2}\right)^{\frac{\varepsilon}{2+\varepsilon}} \cdot \frac{\ln\left(|\mathcal{G}_{\alpha,\beta}|\right) - \ln(\zeta(\beta))}{d + b\beta} \quad .$$

Before giving the proof of the theorem, we will first show that the terms $j^{-\beta_f}$ converge to j^{-2} as $n \to \infty$. More precisely, we show that $j^{-\beta_f} \in \left[\frac{1}{n^{1/f(n)}} \cdot \frac{1}{j^2},\ \frac{1}{j^2}\right]$. First, we give an additive bound for the terms $j^{-\beta_f}$ as follows: $\frac{1}{j^{\beta_f}} = \frac{1}{j^{2+\frac{1}{f(n)}}} \in \left[\frac{1}{j^2} - \tau(n), \frac{1}{j^2}\right]$, where

$$\tau(n) = \max\left\{ \frac{1}{j^2} - \frac{1}{j^{2-\frac{1}{f(n)}}} \,\middle|\, j = 1, \ldots, \Delta_f \right\} = \max\left\{ \frac{j^{\frac{1}{f(n)}} - 1}{j^{2+\frac{1}{f(n)}}} \,\middle|\, j = 1, \ldots, \Delta_f \right\}.$$

We consider the function $x \mapsto h(x) := \frac{x^{1/f(n)}-1}{x^{2+1/f(n)}} = x^{-2} - x^{-2-\frac{1}{f(n)}}$. Its derivative is $\frac{d}{dx}h(x) = \frac{d}{dx}\frac{x^{1/f(n)}-1}{x^{2+1/f(n)}} = -2x^{-3} + \left(2 + \frac{1}{f(n)}\right)x^{-3-\frac{1}{f(n)}}$. The condition $h(x) < 0$ is equivalent to $1 + \frac{1}{2f(n)} < x^{\frac{1}{f(n)}}$. We observe that the derivative attains its maximum at $x = 2$. We have $h'(2) < 0 \iff \left(1 + \frac{1}{2f(n)}\right)^{f(n)} < 2$. We observe that $\lim_{n\to\infty}\left(1 + \frac{1}{2f(n)}\right)^{f(n)} = e^{1/2} < 2$. Thus, we obtain $\tau(n) = \frac{2^{1/f(n)}-1}{2^{2+1/f(n)}}$. Now, we give a multiplicative bound as follows: $\frac{1}{j^{\beta_f}} = \frac{1}{j^2} \cdot j^{2-\beta_f} = \frac{1}{j^2} \cdot \frac{1}{j^{1/f(n)}} \in \left[\frac{1}{n^{1/f(n)}} \cdot \frac{1}{j^2}, \frac{1}{j^2}\right]$.

Let us now give sufficiently precise estimates of sizes and volumes of the node intervals in the functional case.

Lemma 8. *Let $\beta_f = 2 + \frac{1}{f(n)}$ and $X = [x\Delta_f, y\Delta_f]$. We have the following bounds on the size and the volume of the interval:*

$$|[x\Delta_f, y\Delta_f]| \in \left[e^{\alpha\frac{f(n)+1}{2f(n)+1}} \cdot \left(\frac{1}{x} - \frac{1}{y}\right) - (y-x)\Delta_f,\ e^{\alpha\frac{f(n)+1}{2f(n)+1}} \cdot \left(\frac{1}{x} - \frac{1}{y}\right) + e^{\alpha\frac{1}{2f(n)+1}} \cdot \left(\frac{1}{x^2} - \frac{1}{y^2}\right)\right] \quad \textit{and}$$

$$\mathrm{vol}([x\Delta_f, y\Delta_f]) \in \left[\frac{e^{\alpha}(\ln(y) - \ln(x))}{n^{\frac{1}{f(n)}}} - \frac{(y^2 - x^2)\Delta_f^2 + (x+y)\Delta_f}{2},\ e^{\alpha}(\ln(y) - \ln(x)) + e^{\alpha}\left(\frac{1}{x\Delta_f} - \frac{1}{y\Delta_f}\right)\right].$$

Proof. For $\beta = 2$, our technique based on integration yields the following estimate of sizes of intervals:

$$\begin{aligned}\sum_{j=x\Delta}^{y\Delta}\frac{1}{j^2} &\in \left[\int_{x\Delta}^{y\Delta} j^{-2}\,\mathrm{d}j,\ \int_{x\Delta}^{y\Delta} j^{-2}\,\mathrm{d}j + \frac{1}{(x\Delta)^2} - \frac{1}{(y\Delta)^2}\right] \\ &= \left[\frac{1}{x\Delta} - \frac{1}{y\Delta},\ \frac{1}{x\Delta} - \frac{1}{y\Delta} + \frac{1}{(x\Delta)^2} - \frac{1}{(y\Delta)^2}\right] \\ |[x\Delta, y\Delta]| &\in \left[e^{\alpha/2} \cdot \left(\frac{1}{x} - \frac{1}{y}\right),\ e^{\alpha/2} \cdot \left(\frac{1}{x} - \frac{1}{y}\right) + \frac{1}{x^2} - \frac{1}{y^2}\right].\end{aligned}$$

We combine this with the multiplicative bound and obtain the following estimate of the size of intervals in the case $\beta_f = 2 + \frac{1}{f(n)}$.

$$|[x\Delta_f, y\Delta_f]| = \sum_{j=x\Delta_f}^{y\Delta_f} \left\lfloor \frac{e^\alpha}{j^{\beta_f}} \right\rfloor$$

$$\in \left[e^{\alpha \cdot \frac{1+\frac{1}{f(n)}}{2+\frac{1}{f(n)}}} \cdot \left(\frac{1}{x} - \frac{1}{y} \right) - (y-x)\Delta_f,\ e^{\alpha \cdot \frac{1+\frac{1}{f(n)}}{2+\frac{1}{f(n)}}} \cdot \left(\frac{1}{x} - \frac{1}{y} \right) + e^{\alpha \cdot \left(1 - \frac{1}{1+\frac{1}{2f(n)}} \right)} \cdot \left(\frac{1}{x^2} - \frac{1}{y^2} \right) \right]$$

$$= \left[e^{\alpha \cdot \frac{f(n)+1}{2f(n)+1}} \cdot \left(\frac{1}{x} - \frac{1}{y} \right) - (y-x)\Delta_f,\ e^{\alpha \cdot \frac{f(n)+1}{2f(n)+1}} \cdot \left(\frac{1}{x} - \frac{1}{y} \right) + e^{\alpha \cdot \frac{1}{2f(n)+1}} \cdot \left(\frac{1}{x^2} - \frac{1}{y^2} \right) \right].$$

Especially we obtain the following estimate of the size of $\mathcal{G}_{\alpha,\beta_f}$:

$$|[1, \Delta_f]| \in \left[e^\alpha - e^{\alpha \frac{f(n)+1}{2f(n)+1}} - e^{\alpha \frac{f(n)}{2f(n)+1}} + 1,\ e^\alpha - e^{\alpha \frac{f(n)+1}{2f(n)+1}} + e^{\alpha \frac{1}{2f(n)+1}} \cdot e^{2\alpha \frac{f(n)}{2f(n)+1}} - e^{\alpha \frac{1}{2f(n)+1}} \right]$$

$$= \left[(1 - o(1))e^\alpha,\ (2 - o(1))e^\alpha \right].$$

This estimate can be refined as follows:

$$\sum_{j=1}^{\Delta_f} \left\lfloor \frac{e^\alpha}{j^{\beta_f}} \right\rfloor \in \left[\sum_{j=1}^{\Delta_f} \frac{e^\alpha}{j^{\beta_f}} \ - \ \Delta_f,\ \sum_{j=1}^{\Delta_f} \frac{e^\alpha}{j^{\beta_f}} \right]$$

$$\subseteq \left[\frac{1}{n^{1/f(n)}} \cdot \sum_{j=1}^{\Delta_f} \frac{e^\alpha}{j^2} \ - \ \Delta_f,\ \sum_{j=1}^{\Delta_f} \frac{e^\alpha}{j^2} \right] \ \subseteq \ \left[(1 - o(1))\zeta(2)e^\alpha,\ \zeta(2)e^\alpha \right],$$

where the last inclusion holds for $f(n) = \omega(\log(\alpha))$. The volume can be estimated as follows:

$$\operatorname{vol}([x\Delta_f, y\Delta_f]) = \sum_{x\Delta_f}^{y\Delta_f} \left\lfloor \frac{e^\alpha}{j^{\beta_f}} \right\rfloor \cdot j$$

$$\in \left[\sum_{x\Delta_f}^{y\Delta_f} \frac{e^\alpha}{j^{\beta_f - 1}} \ - \ (x\Delta_f + (x\Delta_f + 1) + \ldots + y\Delta_f),\ \sum_{x\Delta_f}^{y\Delta_f} \frac{e^\alpha}{j^{\beta_f - 1}} \right]$$

$$= \left[\sum_{x\Delta_f}^{y\Delta_f} \frac{e^\alpha}{j^{\beta_f - 1}} \ - \ \frac{(y^2 - x^2)\Delta_f^2 \ + \ (x+y)\Delta_f}{2},\ \sum_{x\Delta_f}^{y\Delta_f} \frac{e^\alpha}{j^{\beta_f - 1}} \right].$$

Since $j^{\beta_f - 1} = j^{1 + \frac{1}{f(n)}}, j = x\Delta_f, y\Delta_f$, we use Lemma 23 from our previous paper (cf. [11, p. 23]) and obtain that the volume $\operatorname{vol}([x\Delta_f, y\Delta_f])$ is within the interval

$$\left[\frac{e^\alpha \cdot (\ln(y) - \ln(x))}{n^{\frac{1}{f(n)}}} - \frac{(y^2 - x^2)\Delta_f^2 \ + \ (x+y)\Delta_f}{2},\ e^\alpha \cdot (\ln(y) - \ln(x)) + e^\alpha \cdot \left(\frac{1}{x\Delta_f} - \frac{1}{y\Delta_f} \right) \right].$$

$\square$

We are now prepared to give the proof of Theorem 5.

Proof of Theorem 5. We compute the parameters α, d, x, y of our embedding $G_{U,\mathcal{S}} \mapsto \mathcal{G}_{\alpha,\beta_f}$ for the functional case $\beta_f = 2 + \frac{1}{f(n)}$, $f(n) = \omega(\log(n))$. In order to satisfy constraint (1), we have to give an estimate for $\left|\left[N^{a/d}, N^{b/d}\right]\right|$. Note that $e^{\alpha \cdot \frac{1}{2f(n)-1}} \cdot \Delta_f^2 = e^{\alpha \cdot \frac{f(n)+1}{2f(n)+1}} \cdot \Delta_f = e^{\alpha}$. Thus, our estimate of interval sizes yields

$$\left|\left[N^{a/d}, N^{b/d}\right]\right| \in \left[e^{\alpha}\left(\frac{1}{N^{\frac{a}{d}}} - \frac{1}{N^{\frac{b}{d}}}\right) - \left(N^{\frac{b}{d}} - N^{\frac{a}{d}}\right), e^{\alpha}\left(\frac{1}{N^{\frac{a}{d}}} - \frac{1}{N^{\frac{b}{d}}}\right) + e^{\alpha}\left(\frac{1}{N^{\frac{2a}{d}}} - \frac{1}{N^{\frac{2b}{d}}}\right)\right] \ .$$

In order to satisfy constraint (1), for a given d, we have to choose α such that

$$\left|\left[N^{a/d}\right]\right| \geq e^{\alpha}\left(\frac{1}{N^{\frac{a}{d}}} - \frac{1}{N^{\frac{b}{d}}}\right) - \left(N^{\frac{b}{d}} - N^{\frac{a}{d}}\right) \Longleftrightarrow e^{\alpha}\left(N^{\frac{b-a}{d}} - 1\right) - \left(1 - N^{\frac{a-b}{d}}\right) \geq N^{1+\frac{b}{d}} \ .$$

Hence, we choose

$$e^{\alpha} \approx N^{1+\frac{a}{d}} \iff \alpha \approx \left(1 + \frac{a}{d}\right) \cdot \ln(N) \ .$$

If we now choose $d > \frac{(b+1)\beta_f}{\beta_f - 1}$, then the constraint (2) holds, and for $y = 1$ and $x > 0$ such that $x\Delta_f > N^{b/d}$, constraint (3) holds as well. Thus, we obtain asymptotically the same approximation hardness result as for the case $\beta = 2$ in Theorem 3. □

Now we consider the case when $f(n) = o(\log(n))$. This means that $\beta_f = 2 + 1/f(n)$ is a function of n which slowly converges to 2 from above.

Here the hardness of Min-DS shows a surprising phase transition. We obtain the following theorem.

Theorem 6. *For $\beta_f = 2 + f(n)^{-1}$ with $f(n) = o(\log(n))$, the* Min-DS *problem on (α, β_f)-PLG is in APX.*

Proof. We consider the case when $f(n)$ is a "*slowly growing*" function, namely $f(n) = o(\log(n))$. In that case, $n^{1/f(n)} \longrightarrow \infty$ as $n \longrightarrow \infty$. For $x\Delta_f \leq j \leq y\Delta_f$, we obtain

$$\frac{1}{j^{1+\frac{1}{f(n)}}} = \frac{1}{j} \cdot \frac{1}{j^{\frac{1}{f(n)}}} \leq \frac{1}{j} \cdot \frac{1}{(x\Delta_f)^{\frac{1}{f(n)}}} = \frac{1}{j} \cdot \frac{1}{x^{\frac{1}{f(n)}}} \cdot \frac{1}{e^{\alpha \cdot \frac{1}{2f(n)+1}}} \ ,$$

and therefore

$$\mathrm{vol}([x\Delta_f, \Delta_f]) \leq e^{\alpha} \cdot \ln\left(\frac{1}{x}\right) \cdot \frac{1}{x^{\frac{1}{f(n)}}} \cdot \frac{1}{e^{\alpha \cdot \frac{1}{2f(n)+1}}} \ ,$$

which yields the requirement $\frac{\ln(1/x)}{x^{1/f(n)}} \ \geq \ c \cdot e^{\alpha \cdot \frac{1}{2f(n)+1}}$. This is equivalent to

$$\ln\ln\left(\frac{1}{x}\right) + \frac{1}{f(n)} \cdot \ln\left(\frac{1}{x}\right) \ \geq \ \ln(c) + \frac{\alpha}{2f(n)+1} \ ,$$

which means the following: In order to dominate the remaining vertices of the graph with vertices from $[x\Delta_f, \Delta_f]$, we have to choose (roughly) $\ln({}^1\!/x) \geq {}^{\alpha}\!/2$, i. e. ${}^1\!/x \geq e^{\alpha/2}$. This gives the following lower bound for the size of that interval:

$$|[x\Delta_f, \Delta_f]| \;\geq\; e^{\alpha \cdot \frac{f(n)+1}{2f(n)+1}} \cdot \left(e^{\frac{\alpha}{2}} - 1\right) - \left(1 - \frac{1}{e^{\frac{\alpha}{2}}}\right) \cdot e^{\frac{\alpha}{2+\frac{1}{f(n)}}} \;\geq\; (1-o(1))e^{\frac{\alpha}{2}\cdot\left(1+\frac{f(n)+1}{f(n)+1/2}\right)} \;.$$

This lower bound for the size of $[x\Delta_f, \Delta_f]$ converges to e^{α} as $n \to \infty$, which means there exists some $c > 0$ such that $|[x\Delta_f, \Delta_f]| \geq c \cdot |\mathcal{G}_{\alpha,\beta_f}|$ in order to be a dominating set. Hence, each dominating set in $\mathcal{G}_{\alpha,\beta_f}$ is of cardinality at least $c \cdot |\mathcal{G}_{\alpha,\beta_f}|$ and we obtain the result. $\square$

10 Further Research

The further improvements on both lower and upper approximation bounds are important open questions in the area, especially the upper approximation bounds for $\beta \leq 2$. Another interesting problem concerns the approximability of PLG optimization problems on random or quasirandom instances.

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