Optimal Cuts and Bisections on the Real Line in Polynomial Time

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Abstract

The exact complexity of geometric cuts and bisections is the longstanding open problem including even the dimension one. In this paper, we resolve this problem for dimension one (the real line) by designing an exact polynomial time algorithm. Our results depend on a new technique of dealing with metric equalities and their connection to dynamic programming. The method of our solution could be also of independent interest.

1 Introduction

The metric MAX-CUT, MAX-BISECTION, MIN-BISECTION and other Partitioning problems were all proved to have *polynomial time approximation schemes* (PTAS) [5, 3, 1, 4, 2, 6]. The above problems are known to be NP-hard in exact setting. The status of those problems for geometric (thus including Euclidean) metrics and this even for dimension one was widely open.

In this paper we resolve the status of those problems for just dimension one by giving a polynomial time algorithm. Our solution, somewhat surprisingly, involves certain new ideas for applying dynamic programming which could be also of independent interest.

2 Preliminaries and General Setting

We shall define our dynamic programming method in terms of generalized subproblems on finite multisets of reals generalizing slightly a geometric metric setting.

For a partition of a finite multiset P of reals into two multisets P_1 and P_2, the *value of the cut* is the total length of all intervals on the real line that have one endpoint in P_1 and the other one in P_2.

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The MAX-CUT problem for P will be now to find a partition of P into two multisets that maximizes the value of the cut. If $|P| = n$ and the two multisets are additionally required to be of cardinality k and $n-k$, respectively, then we obtain the $(k, n-k)$ MAX-PARTITION problem for P. In particular, if n is even and $k = n/2$ then we have the MAX-BISECTION problem. Next, if we replace the requirement of maximization with that of minimization then we obtain the $(k, n-k)$ MIN-PARTITION and MIN-BISECTION problems for P, respectively.

In this paper we study geometric instances of the above problems in dimension one (the real line) which could be rephrased as the problems of partitioning arbitrary finite metric spaces of this dimension.

3 The Algorithm

The global idea behind our algorithm is as follows. We guess how many of the copies of the rightmost real are respectively in the first and second set and move them to the next to the rightmost real. We also guess how many copies of the reals in the remaining part of the multiset are in the first and second set respectively. Having this information, we can compute exactly the difference between the value of optimal solution for the whole input multiset and that for the multiset resulting from the movement under the guessed partition proportions. The difference can be easily evaluated due to the triangle equality that holds on the real line. Now, we solve the problem for the transformed multiset whose elements are copies of the shrunk set of reals recursively under the guessed partition proportions, after additionally guessing the proportions of the partition for the original copies of the next to the right real. Eventually, we end up with the trivial multiset where all elements are copies of the leftmost real. We eliminate guesses by an enumeration of all possibilities and choosing the best one. Next, we solve the resulting subproblems in bottom up fashion instead the top down to obtain a polynomial time solution.

Consider a finite multiset P of reals. We assume that $|P| = n$ and that P consists of copies of $l \leq n$ distinct reals. For $i = 1, ..., l$, let x_i denote the i-th smallest real whose copy is in P, and let P_i denote the sub-multiset of P consisting of all elements of P which are copies of reals in $\{x_1, ..., x_i\}$. For convention, we assume $P_0 = \emptyset$.

We shall consider a family of generalized subproblems $S_i(p, q, r, t)$, where $i \in \{1, ..., l\}$, p, q, r, t are integers in $\{0, ..., n\}$ such that $p + q = |P_{i-1}|$ and $r + t = n - |P_{i-1}|$. The subproblem $S_i(p, q, r, t)$ is to find for a multiset that is the union of P_{i-1} with $r + t$ copies of x_i a partition into two multisets such that p elements of P_i and r copies of x_i form the first set and the value of the cut is maximized. The value of such a maximum cut is denoted by $MAXCUT(S_i(p, q, r, t))$.

Lemma 1. *For* $i \geq 2$, $MAXCUT(S_i(p, q, r, t)) = (x_i - x_{i-1})(pt + qr) +$
$+ \max_{r_0 \leq p \wedge t_0 \leq q \wedge r_0 + t_0 = |P_{i-1}| - |P_{i-2}|} MAXCUT(S_{i-1}(p - r_0, q - t_0, r_0 + r, t_0 + t))$.

Proof. For $1 \leq r_0 \leq p \wedge 1 \leq t_0 \leq q$, consider an optimal solution to $S_{i-1}(p - r_0, q - t_0, r_0 + r, t_0 + t)$, where $r_0 + t_0 = |P_{i-1}| - |P_{i-2}|$. Let us move r copies of the real x_{i-1} in the first

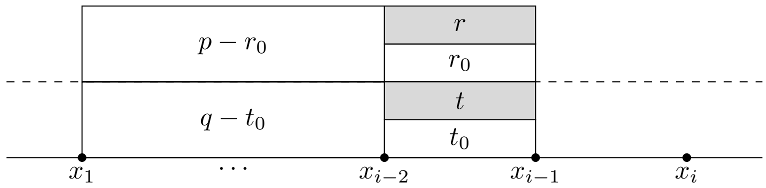

Figure 1: $S_{i-1}(p - r_0, q - t_0, r_0 + r, t_0 + t)$.

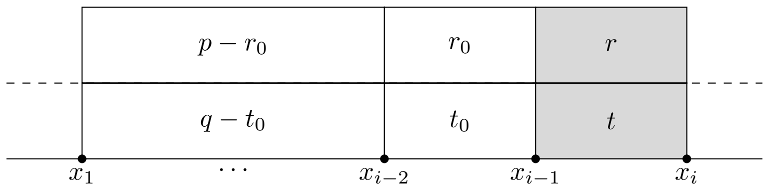

Figure 2: $S_i(p, q, r, t)$.

set of the solution to the real x_i and t copies of x_{i-1} in the second set of the solution to the same real x_i. We obtain a feasible solution to $S_i(p, q, r, t)$ whose cut value is $(x_i - x_{i-1})(pt + qr) + MAXCUT(S_{i-1}(p - r_0, q - t_0, r_0 + r, t_0 + t))$. It follows that $MAXCUT(S_i(p, q, r, t)) \geq (x_i - x_{i-1})(pt + qr) +$
$+ \max_{r_0 \leq p \wedge t_0 \leq q \wedge r_0 + t_0 = |P_{i-1}| - |P_{i-2}|} MAXCUT(S_{i-1}(p - r_0, q - t_0, r_0 + r, t_0 + t))$.

Contrary, consider an optimal solution to $S_i(p, q, r, t)$. Suppose that r_0 copies of x_{i-1} are in the first set of the solution and t_0 copies of x_{i-1} are in the second set of the solution. Note that $r_0 + t_0 = |P_{i-1}| - |P_{i-2}|$ holds. Let us move r copies of x_i in the first set of the solution to the real x_{i-1} and t copies of x_i in the second set of the solution to the same real x_{i-1}. We obtain a feasible solution to $S_{i-1}(p - r_0, q - t_0, r_0 + r, t_0 + t)$ whose cut value is $MAXCUT(S_i(p, q, r, t)) - (x_i - x_{i-1})(pt + qr)$. It follows that $MAXCUT(S_i(p, q, r, t)) \leq (x_i - x_{i-1})(pt + qr) +$
$+ \max_{r_0 \leq p \wedge t_0 \leq q \wedge r_0 + t_0 = |P_{i-1}| - |P_{i-2}|} MAXCUT(S_{i-1}(p - r_0, q - t_0, r_0 + r, t_0 + t))$. $\square$

Theorem 1. *The geometric MAX-CUT problem on the real line as well as the geometric MAX-BISECTION problem on the real line are solvable in $O(n^4)$ time.*

Proof. First, we shall show that the subproblems $S_i(p, q, r, t)$, where $i \in \{1, ..., l\}$, p, q, r, t are integers in $\{0, ..., n\}$ such that $p + q = |P_{i-1}|$ and $r + t = n - |P_{i-1}|$ are solvable in $O(n^4)$ time.

Since $p + q = |P_{i-1}|$, there are $O(n)$ choices for the parameters p, q, and similarly since $r + t = n - |P_{i-1}|$, there are $O(n)$ choices for the parameters r, t. It follows that the total number of considered subproblems is $O(n^3)$.

We can compute the values of $MAXCUT(S_i(p, q, r, t))$ in bottom up fashion in increasing i order. If $i = 1$, then $p = 0$ and $q = 0$ and $MAXCUT(S_i(p, q, r, t)) = 0$ holds trivially. For $i \geq 2$, we apply Lemma 1 in order to compute the value of $MAXCUT(S_i(p, q, r, t))$ in $O(n)$

time by $r_0 + t_0 = |P_{i-1}| - |P_{i-2}|$. The corresponding optimal solutions can be obtained by backtracking. The upper bound $O(n^4)$ follows.

Let x_l be the largest real whose copy is in P, and let m be the number of copies of x_l in P. The optimal solution to the geometric MAX-CUT problem for P can be found among the optimal solutions to the $O(n^2)$ subproblems $S_k(p, q, r, t)$, where $p+q = |P_{k-1}|$ and $r+t = m$. Furthermore, the optimal solution to the geometric $(k, n-k)$ MIN-PARTITION problem for P can be found among the optimal solutions to the $O(n)$ subproblems $S_k(p, q, r, t)$, where $p + q = |P_{k-1}|$, $r + t = m$, $p + r = k$ and $q + t = n - k$. □

To solve the $(k, n-k)$ MIN-PARTITION problem on the real line, we consider an analogous family of generalized subproblems $U_i(p, q, r, t)$, where $i \in \{1, ..., l\}$, p, q, r, t are integers in $\{0, ..., n\}$ such that $p + q = |P_{i-1}|$ and $r + t = n - |P_{i-1}|$. The subproblem $U_i(p, q, r, t)$ is to find for a multiset that is the union of P_{i-1} with $r + t$ copies of x_i a partition into two multisets such that p elements of P_i and r copies of x_i form the first set and the value of the cut is minimized. The value of such a minimum cut is denoted by $MINCUT(S_i(p, q, r, t))$.

Analogously, we obtain the following counterparts of Lemma 1 and Theorem 1 for $(k, n-k)$ MIN-PARTITION and MIN-BISECTION.

Lemma 2. *For $i \geq 2$, $MINCUT(U_i(p, q, r, t)) =$
$(x_i - x_{i-1})(pt + qr) + \min_{r_0 \leq p \wedge t_0 \leq q \wedge r_0 + t_0 = |P_{i-1}| - |P_{i-2}|} MINCUT(U_{i-1}(p - r_0, q - t_0, r_0 + r, t_0 + t))$.*

Theorem 2. *The geometric $(k, n-k)$ MIN-PARTITION problem on the real line, in particular the geometric MIN-BISECTION problem on the real line, are analogously solvable by dynamic programming in $O(n^4)$ time.*

4 Final Remarks

It remains an open problem whether our method can be generalized to higher dimensions or those problems turn out to be inherently hard. At stake is the exact computational status of other geometric problems for which our knowledge is very limited at the moment.

5 Acknowledgments

We thank Uri Feige, Ravi Kannan and Christos Levcopoulos for a number of interesting discussions on the subject of this paper.

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