Approximability of the Vertex Cover Problem in Power Law Graphs

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Abstract

In this paper we construct an approximation algorithm for the Minimum Vertex Cover Problem (Min-VC) with an expected approximation ratio of $2 - \frac{\zeta(\beta)-1-\frac{1}{2^\beta}}{2^\beta\zeta(\beta-1)\zeta(\beta)}$ for random *Power Law Graphs* (PLG) in the (α, β)-model of Aiello et. al.. We obtain this result by combining the Nemhauser and Trotter approach for Min-VC with a new deterministic rounding procedure which achieves an approximation ratio of $\frac{3}{2}$ on a subset of low degree vertices for which the expected contribution to the cost of the associated linear program is sufficiently large.

1 Introduction

In recent years topological analyses have been applied to a variety of real world graphs such as the World-Wide Web, the Internet, Collaboration and Social Networks, Protein Interaction Networks and other large-scale graphs of biological systems. Typical statistical parameters such as the *diameter*, *robustness*, *clustering coefficient* and *degree distribution* have been measured and compared to the expected values of these parameters in *uniform* random graph models such as the classical $G(n, p)$-Model due to Erdős and Rényi [ER60]. It turned out that the real world graphs are significantly different from the random models with respect to these statistical and topological properties. In subsequent studies the aim was to describe the properties of real world networks mathematically and to propose new models in order to meet these conditions.

As of 1999 Kumar et. al. [BKM+00, KRR+00], Kleinberg et. al. [KKR+99, KL01] and Faloutsos, Faloutsos and Faloutsos [FFF99, SFFF03] measured the degree sequence of the World-Wide Web and independently observed that it is well approximated by a power law distribution, i.e. the number of nodes y_i of a given degree i is proportional to $i^{-\beta}$ where $\beta > 0$. This was later verified for a large number of existing real-world networks such as protein-protein interactions, gene regulatory networks, peer-to-peer networks, mobile call networks and social networks [JAB01, GBBK02, SMS+08, EKM+04].

In order to analyze these graphs, some research has been directed towards finding suitable models for describing structural properties quantitatively and qualitatively. A number of *Power Law Graph* (PLG) models have been proposed, such as the Barabási-Albert model of *Preferential Attachment* [BA99], the *Buckley-Osthus Model* [BO04], the *Cooper-Frieze Model* [CF03] and the *Copying Model* due to Kumar et. al. [KRR+00]. All these models describe a *random growth process* starting from a small seed graph and yielding – besides other features – a power law degree sequences in the limit.

A different approach is to take a power law degree sequence as input and to generate a graph instance with this distribution in a random fashion. Among the most widely known models of

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this kind is the *ACL-Model* due to Aiello, Chung and Lu [ACL01]. Here, the number y_i of vertices with degree i is roughly given by $y_i \approx e^\alpha / i^\beta$, where e^α is a normalization constant which determines the size of the graph. While this model is potentially less accurate than the detailed description of a growth process, it has the advantage of being robust and general, i.e., structural properties that are true in this model will be true for the majority of graphs with the given degree sequence.

All of the above models are well motivated and there exists a large body of literature on mathematical foundations and applications [BA99, ACL00, BR02, EKM$^+$04, MPS06]. In this paper, we focus on the ACL-Model for random PLG which we will refer to as the (α, β)-*Model*.

Apart from having certain structural properties, such as high clustering coefficient, small-world characteristics and self similarity, there exists practical evidence that combinatorial optimization in PLG is easier than in general graphs [PL01, GMS03, EKM$^+$04, KGS06]. Contrasting this Ferrante et. al. [FPP08] and Shen et. al. [SNT10] studied the approximation hardness of certain optimization problems in *combinatorial Power Law Graphs* and showed NP-hardness and APX-hardness of classical problems such as Minimum Vertex Cover (Min-VC), Maximum Independent Set (Max-IS) and Minimum Dominating Set (Min-DS). In this paper we study the approximability of the Minimum Vertex Cover problem in the random Power Law Graph model of Aiello et. al. [ACL01].

The Minimum Vertex Cover is one of the most well-studied problems in combinatorial optimization. A *vertex cover* of a graph $G = (V, E)$ is a set of vertices $C \subseteq V$ such that each edge $e = \{u, v\}$ of G has at least one endpoint in C. The Minimum Vertex Cover problem (Min-VC) is the the problem of finding a cover of minimum cardinality in a graph. The problem is known to be NP-complete due to Karp's original proof [Kar72] and APX-complete [PY91]. Moreover, it cannot be approximated within a factor of 1.3606 [DS05], unless P = NP, and is inapproximable within $2 - \epsilon$ for any $\epsilon > 0$ as long as the Unique Games Conjecture (UGC) holds true [KR08]. Here, we show that the Min-VC problem can be approximated with an expected approximation ratio < 2 in random Power Law Graphs:

Theorem 1. *There exists a polynomial time algorithm which approximates the* Minimum Vertex Cover *problem (*Min-VC*) in random Power Law Graphs in the* (α, β)*-Model for* $\beta > 2$ *(where graphs are given instance by instance) with an expected approximation ratio of*

$$\rho = 2 - \frac{\zeta(\beta) - 1 - \frac{1}{2^\beta}}{2^\beta \zeta(\beta - 1)\zeta(\beta)}.$$

We also give a refined analysis for the case $\beta > 2.424$ and obtain the following improvement.

Theorem 2. *For* $\beta > 2.424$*, the* Minimum Vertex Cover *problem (*Min-VC*) in the* (α, β)*-Model can be approximated with expected asymptotic approximation ratio*

$$\rho' = 2 - \frac{\left(\zeta(\beta) - 1 - \frac{1}{2^\beta}\right) \cdot \zeta(\beta - 1)}{\zeta(\beta - 1) \cdot \zeta(\beta)} \left[1 - \left(\frac{\zeta(\beta - 1) - \left(1 + \frac{1}{2^{\beta - 1}}\right)}{\zeta(\beta - 1)}\right)^3\right].$$

In Figure 1 these two upper bounds ρ and ρ' are shown as functions of the parameter β.

The paper is organized as follows. In subsection 2.1 we describe the (α, β)-model for Power Law Graphs, describe the random generation process and give a formal description of the model parameters. In subsection 2.2 we give some background on the Min-VC problem and briefly describe the *half-integral solution* method proposed by Nemhauser and Trotter. Section 3 presents our new approximation algorithm for Min-VC in Power Law Graphs. This algorithm basically consists of a deterministic rounding procedure on a half-integral solution for Min-VC.

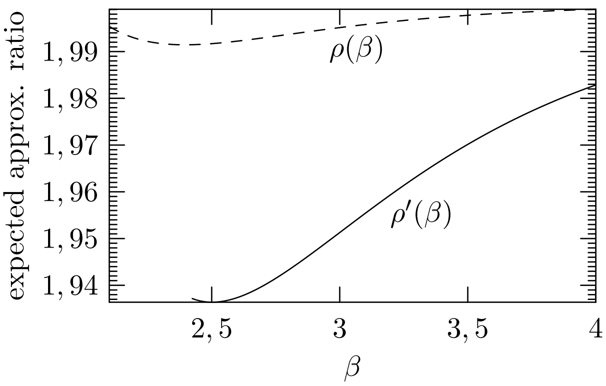


Figure 1: Comparison of first (- - -) and second (——) analysis in terms of functions of the parameter β, for $\beta > 2$ and $\beta > 2.424$, respectively.

In Section 3.1 we show that this rounding procedure yields an approximation ratio of $\frac{3}{2}$ in the subgraph induced by the low-degree vertices of the Power Law Graph and a 2-approximation in the residual graph. In Section 3.2 we construct upper and lower bounds on the expected size of the half-integral solution in the induced subgraph of low-degree vertices and finally prove our main theorems. We conclude the paper by giving a short summary and further research in Section 4.

2 Preliminaries

2.1 (α, β)-Power Law Graphs

In this section we describe the random PLG-model proposed by Aiello, Chung and Lu in [ACL01], which we will denote as $M(\alpha, \beta)$. This model considers a random graph with the following degree distribution depending on two given values α and β: For each $1 \leqslant i \leqslant \Delta = \left\lfloor e^{\frac{\alpha}{\beta}} \right\rfloor$ there are y_i vertices of degree i with

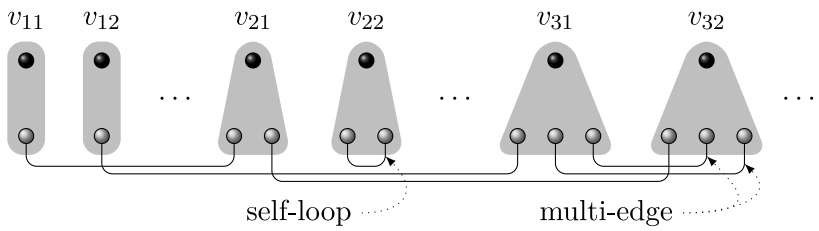
$$y_i = \begin{cases} \left\lfloor \frac{e^\alpha}{i^\beta} \right\rfloor & \text{if } i > 1 \text{ or } \sum_{i=1}^{\Delta} \left\lfloor \frac{e^\alpha}{i^\beta} \right\rfloor \text{ is even} \\ \lfloor e^\alpha \rfloor + 1 & \text{otherwise.} \end{cases}$$

Here, i and y_i satisfy $\log y_i = \alpha - \beta \log i$. The variable α is the logarithm of the size of the graph and β is the log-log growth rate. Let $\mathcal{G}_{(\alpha,\beta)}$ be the set of all undirected graphs with multi-edges and self-loops on $n = \sum_{i=1}^{\Delta} y_i$ vertices which have y_i vertices of degree i ($1 \leqslant i \leqslant \Delta$). Then $M(\alpha, \beta)$ is the distribution on $\mathcal{G}_{(\alpha,\beta)}$ obtained in the following way [ACL01]:

1. Generate a set L of $d(v)$ distinct copies of each vertex v.

2. Generate a random matching on the elements of L.

3. For each pair of vertices u and v, the number of edges joining u and v in G is equal to the number of edges in the matching of L which join copies of u to copies of v.



As in [ACL01], in the following we will work with the real numbers $\frac{e^\alpha}{i^\beta}$, $e^{\frac{\alpha}{\beta}}$ instead of their integer counterparts. For $\beta > 2$ the error is a lower order term (c.f. [ACL01], remark on page 6).

A graph $G \in \mathcal{G}_{(\alpha,\beta)}$ has the following properties: The maximum degree of G is $e^{\frac{\alpha}{\beta}}$, and for $\beta > 2$ the number of vertices is $n = \sum_{i=1}^{e^{\alpha/\beta}} \frac{e^\alpha}{i^\beta} \approx \zeta(\beta)e^\alpha$ and the number of edges is $m = \frac{1}{2}\sum_{i=1}^{e^{\alpha/\beta}} i\frac{e^\alpha}{i^\beta} \approx \frac{1}{2}\zeta(\beta-1)e^\alpha$ where the error terms are $o(n)$ and $o(m)$, respectively.

2.2 LP-Relaxation and Half-Integral Solution for Min-VC

In this section we give a brief outline of the *Nemhauser-Trotter Theorem* stated in [NT75] and show how this is used to approximate Min-VC in a graph $G = (V, E)$ as described by Hochbaum et. al. in [HMNT93].

Nemhauser and Trotter considered the following LP-relaxation, which applies to the more general *weighted* vertex cover problem:

$$\begin{array}{lll} \text{minimize} & \displaystyle\sum_{i=1}^{n} w_i x_i, & \\ \text{subject to} & x_i + x_j \geqslant 1, & \text{for each edge } \{v_i, v_j\} \in E, \\ & x_i \geqslant 0, & \text{for each vertex } v_i \in V, \end{array}$$

They show that there always exists an optimal solution x for this LP which is *half-integral*, i.e. for all i, $x_i \in \left\{0, \frac{1}{2}, 1\right\}$. Then they partition the set of vertices into subsets $P, Q, R \subseteq V$, such that $v_i \in P$ if $x_i = 1$, $v_i \in Q$ if $x_i = \frac{1}{2}$ and $v_i \in R$ if $x_i = 0$ in this solution. They show that at least one optimal vertex cover in G contains the set P, that each vertex in R has all its neighbors in P and – moreover – that each cover in G has weight at least $\mathrm{w}(P) + \frac{1}{2}\mathrm{w}(Q)$. From this it follows that at least one optimal vertex cover in G consists of the set P and an optimal cover in the subgraph $H[Q]$ induced by Q.

Hochbaum et. al. [HMNT93] showed that an integer solution y obtained by setting $y_i = 1$ for all vertices $v_i \in Q \cup P$ and $y_i = 0$ for all $v_i \in R$ is a 2-approximate solution for the Min-VC problem in G. Our approximation algorithm for Min-VC in random Power Law Graphs will make use of a half-integral solution x of the LP-relaxation along with the properties described in the Nemhauser-Trotter Theorem in order to achieve an approximation ratio strictly less than 2.

3 Approximation of Min-VC in (α, β)-PLG

In this section we present our main result, namely an approximation algorithm with expected approximation ratio $2 - \frac{\zeta(\beta)-1-\frac{1}{2^\beta}}{2^\beta\zeta(\beta-1)\zeta(\beta)}$ for the Min-VC problem in (α,β)-PLG for $\beta > 2$. Furthermore a refined analysis yields an improved *asymptotic* approximation ratio for the case $\beta > 2.424$.

Let us first give an outline of this algorithm. On instance $G \in \mathcal{G}_{(\alpha,\beta)}$ the algorithm starts with a half-integral solution $x : V \to \left\{0, \frac{1}{2}, 1\right\}$ of the associated LP and uses some deterministic rounding procedure to generate an integral solution $y : V \to \{0,1\}$. We show that for the set $V^* = \bigcup_{v:d(v)\in\{1,2\}} (\{v\} \cup \mathcal{N}(v))$ of degree-1 and degree-2 nodes and their neighbors in G, the rounding procedure satisfies $y(V^*) \leqslant \frac{3}{2} \cdot x(V^*)$ and furthermore $x(V^*)$ is sufficiently large (in expectation) with respect to $M(\alpha,\beta)$.

3.1 Approximation Algorithm

Now, we describe our deterministic rounding procedure (Algorithm 1) on $G = (V,E)$ for $G \in \mathcal{G}_{(\alpha,\beta)}$. First, the algorithm processes all nodes of the subset $V' = L \cup \mathcal{N}(L)$ where $L = \left\{v \in V \middle| \left(d(v) = 2, x(v) = \frac{1}{2}\right) \vee (d(v) = 1)\right\}$ and provides a rounded integral solution y with $y(V') \leqslant \frac{3}{2} \cdot x(V')$. Furthermore we show that $y(V^* \setminus V') \leqslant \frac{4}{3} \cdot x(V^* \setminus V')$ and $y(V \setminus V^*) \leqslant 2 \cdot x(V \setminus V^*)$.

An analysis of the algorithm is provided by the following Lemma 3 and Lemma 4.

Lemma 3. *The assignment y generated by Algorithm 1 is an integer solution and satisfies $y(u) = 1$ for all $u \in V'$ with $d(u) \geqslant 3$.*

Proof. Any high-degree neighbor of degree-1 vertices is set to 1 in step **(1)** of the algorithm.

Since either step **(3)** or **(4)** is processing every single degree-2 vertex $v \in V$ with $x(v) = \frac{1}{2}$, there are no leftover vertices $v \in V'$ of degree 2 with fractional values.

Assume that there is a vertex $u \in V', d(u) \geqslant 3$ and $x(u) = y(u) = \frac{1}{2}$. Then u has at least one degree 2 neighbor v_1 with $x(v_1) = \frac{1}{2}$. Because of step **(3)** and **(4)** of the algorithm, v_1 must have been processed by another degree 2 vertex v_2, setting $y(v_1) = 1$. This again introduces another neighbor w of v_2 with $y(w) = 1$ and leads to the situation of a path uv_1v_2w described in step **(2)**. In this case, the algorithm sets $y(u) = 1$ and thus we have a contradiction to the above assumption. □

Lemma 4. *The assignment y generated by Algorithm 1 satisfies $y(V^*) \leqslant \frac{3}{2} \cdot x(V^*)$.*

Proof. The algorithm partitions the graph induced by V^* into edge-disjoint subgraphs, namely stars whose leaves are degree-1 vertices and paths of length $\leqslant 4$ whose internal nodes are degree-2 vertices. We show that for each such subgraph P_i, $y(P_i) \leqslant \frac{3}{2} \cdot x(P_i)$ and furthermore $y(v) = 1$ for each $v \in V^*$ which is contained in more than one such subgraph.

In step **(1)** of the algorithm all degree-1 vertices and their neighbors are processed.

In step **(2)** the subgraphs are `unprocessed` paths P_i of length 3. Since $P_i = \underset{u}{\bullet}\!-\!\underset{v_1}{\bullet}\!-\!\underset{v_2}{\bullet}\!-\!\underset{w}{\bullet}$ contains two disjoint edges $\{u,v_1\},\{v_2,w\}$, $x(P_i) \geqslant 2$ and particularly $x(v_2) + x(w) \geqslant 1$. Therefore $y(P_i) = 3 \leqslant \frac{3}{2} \cdot x(P_i)$ holds via mapping $\overset{\geqslant 1/2}{\bullet}\!-\!\overset{1/2}{\bullet}\!-\!\overset{\geqslant 1/2}{\bullet}\!-\!\overset{\geqslant 0}{\bullet} \mapsto \overset{1}{\circ}\!-\!\overset{1}{\circ}\!-\!\overset{0}{\circ}\!-\!\overset{1}{\circ}$ (where the gray color indicates a `processed` vertex) and y restricted to P_i (denoted as $y \restriction P_i$) is a vertex cover for P_i.

In step **(3)** all paths $P_i = \underset{u}{\bullet}\!-\!\underset{v}{\bullet}\!-\!\underset{w}{\bullet}$ are processed, where at least one of u, w is of degree $\geqslant 3$. In cases **(3.1)**-**(3.4)** the algorithm considers all possible combinations of some of these nodes being already processed.

In case **(3.1)** u is marked `unprocessed`, w is already `processed` and $x(u) \geqslant \frac{1}{2}$. The rounding algorithm sets $y(v) = 0$ and $y(u) = 1$, mapping $\overset{\geqslant 1/2}{\bullet}\!-\!\overset{1/2}{\bullet}\!-\!\overset{1}{\circ} \mapsto \overset{1}{\circ}\!-\!\overset{0}{\circ}\!-\!\overset{1}{\circ}$, again yielding a vertex cover $y \restriction P_i$ for P_i with $y(P_i) \leqslant x(P_i)$.

In case **(3.2)** we have that both u, w are marked as `unprocessed` and since $x(v) = \frac{1}{2}$ we have that $x(u) \geqslant \frac{1}{2}$ and $x(w) \geqslant \frac{1}{2}$. The rounding algorithm sets $y(v) = 0, y(u) = y(w) = 1$, mapping $\overset{\geqslant 1/2}{\bullet}\!-\!\overset{1/2}{\bullet}\!-\!\overset{\geqslant 1/2}{\bullet} \mapsto \overset{1}{\circ}\!-\!\overset{0}{\circ}\!-\!\overset{1}{\circ}$, and since $x(u) \geqslant \frac{1}{2}$ and $x(w) \geqslant \frac{1}{2}$ we have that $y(P_i) \leqslant \frac{4}{3} \cdot x(P_i)$.

Algorithm 1: DETERMINISTIC ROUNDING

Input: $G = (V, E), x : V \to \left\{0, \frac{1}{2}, 1\right\}$.
Output: $y : V \to \{0, 1\}$.

forall the $v \in V$ **do**
  $y(v) := x(v)$;
  mark v as `unprocessed`;

(0) `compute` $G' = (V', E')$ induced by $V' = L \cup \mathcal{N}(L)$ where
$L = \left\{v \in V \mid \left(d(v) = 2, x(v) = \frac{1}{2}\right) \vee (d(v) = 1)\right\}$;

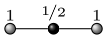
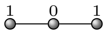
(1) **forall the** $v \in V$ *with* $d(v) = 1$ **do**
  let u be the neighbor of v in G;
  `set` $y(v) = 0$; `set` $y(u) = 1$;

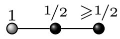
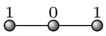
(2) **forall the** $P = uv_1v_2w \subset G'$ `unprocessed`, $d(u) \geqslant 3, d(v_1) = d(v_2) = 2$ **do**
  `set` $y(u) = y(w) = y(v_1) = 1$;
  `set` $y(v_2) = 0$;

(3) **forall the** $v \in V'$ `unprocessed`, $d(v) = 2 \wedge \exists u \in \mathcal{N}(v), d(u) \geqslant 3$ **do**
(3.1) **else if** u `unprocessed`, w `processed` **then**
    `set` $y(v) = 0$; `set` $y(u) = 1$;
(3.2) **else if** *both* u, w `unprocessed` **then**
    `set` $y(v) = 0$; `set` $y(u) = y(w) = 1$; `/* with` $x(u) \geqslant \frac{1}{2}$ `and` $x(w) \geqslant \frac{1}{2}$ `*/`
(3.3) **if** *both* u, w `processed` **then**
    `set` $y(v) = 0$;
(3.4) **else if** u `processed`, w `unprocessed` **then**
    `set` $y(v) = 0$; `set` $y(w) = 1$; `/*` $y(u) = 1$ `already set and` $x(w) \geqslant \frac{1}{2}$ `*/`

(4) **forall the** $v \in V'$ `unprocessed`, $d(v) = 2$ **do**
(4.1) **else if** u `unprocessed`, w `processed` **then**
    `set` $y(v) = 0$; `set` $y(u) = 1$;
(4.2) **else if** *both* u, w `unprocessed` **then**
    `set` $y(v) = 0$; `set` $y(u) = y(w) = 1$; `/* with` $x(u) \geqslant \frac{1}{2}$ `and` $x(w) \geqslant \frac{1}{2}$ `*/`
(4.3) **if** *both* u, w `processed` **then**
    `set` $y(v) = 0$;
(4.4) **else if** u `processed`, w `unprocessed` **then**
    `set` $y(v) = 0$; `set` $y(w) = 1$; `/*` $y(u) = 1$ `already set and` $x(w) \geqslant \frac{1}{2}$ `*/`

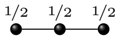
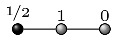
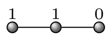
(5) **forall the** $v \in V$ **do**
  if $x(v) = \frac{1}{2}$ **then**
    `set` $y(v) = 1$; `/*` $y(v) = \min\{1, 2 \cdot x(v)\}$ `*/`

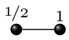
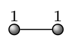
In case **(3.3)** both u, w are marked as `processed` and therefore $y(u) = y(w) = 1$, since u, w are adjacent to processed degree one or degree two vertices other than v. The algorithm sets $y(v) = 0$, mapping  $\mapsto$ . Hence $y \restriction P_i$ is a vertex cover for P_i with $y(P_i) \leqslant x(P_i)$.

In case **(3.4)** u is already `processed` and w is still marked `unprocessed`. Since $x(v) = \frac{1}{2}$ we have that $x(w) \geqslant \frac{1}{2}$. The rounding algorithm sets $y(v) = 0$ and $y(w) = 1$, mapping  $\mapsto$ , and since $x(w) \geqslant \frac{1}{2}$ it yields a vertex cover $y \restriction P_i$ for P_i with $y(P_i) \leqslant x(P_i)$.

Step **(4)** considers all remaining unprocessed vertices of degree 2. If v is such a vertex with neighborhood $\mathcal{N}(v) = \{u, w\}$, the sub-cases **(4.1)**-**(4.4)** are treated analogously to cases **(3.1)**-**(3.4)** and the mapping $x \mapsto y$ achieves $y(P_i) \leqslant \frac{4}{3} \cdot x(P_i)$ on the considered paths P_i.

After steps **(0)**-**(4)** of the algorithm there may still be some remaining high-degree vertices $u \in V^*, d(u) \geqslant 3$ with $x(u) = y(u) = \frac{1}{2}$. These are treated separately (and rounded to $y(u) = 1$ together with all other vertices in $V \setminus (V' \setminus V^*)$) in step **(5)** of the algorithm. We have to argue that $y(V^*) \leqslant \frac{3}{2} \cdot x(V^*)$ still holds true.

We consider first the case that $u \in V', d(u) \geqslant 3$ and $x(u) = y(u) = \frac{1}{2}$. Then u has a neighbor v of degree $\leqslant 2$ with $x(v) = \frac{1}{2}$ and $y(v) = 1$, and since $y(u) = \frac{1}{2}$ we have $d(v) = 2$. Let v_2 be the other neighbor of v, then $d(v_2) = 1$ (since otherwise the second neighbor w of v_2 would give rise to a path of length 3, containing also u and hence would have been processed in step **(2)**). But then locally on the set $\{u, v, v_2\}$ we have the mapping  $\mapsto$  $\mapsto$  with a local ratio of $\frac{4}{3}$.

Let us now assume $u \in V^* \setminus V', d(u) \geqslant 3$ and $x(u) = y(u) = \frac{1}{2}$. Then every degree-2 neighbor v has $x(v) \neq \frac{1}{2}$, hence $x(v) = 1$, and therefore $y(v) = 1$. We show that $v \notin V'$, i.e. that v was not processed by the algorithm and can be treated as a part of a subgraph disjoint to G' in G. Let $w \in \mathcal{N}(v)$ be the second neighbor of v besides u. Then $x(w) = 0$ since otherwise (in case $x(w) \geqslant \frac{1}{2}$) we could decrease $x(v)$ from 1 to $\frac{1}{2}$ and still have a feasible half-integral solution, which would contradict the optimality of x. Therefore $v, w \notin V'$, which means that v, w are not processed by the algorithm. Rounding $y(u) = 1$, mapping  $\mapsto$ , yields a vertex cover $y \restriction \{u, v\}$ with $y(\{u, v\}) \leqslant \frac{4}{3} \cdot x(\{u, v\})$.

We conclude that the assignment $y : V \mapsto \{0, 1\}$ is a vertex cover of G with $y(V^*) \leqslant \frac{3}{2} \cdot x(V^*)$ and $y(V \setminus V^*) \leqslant 2 \cdot x(V \setminus V^*)$. ☐

3.2 Expected Approximation Ratio

The following lemma shows how to retrieve an expected approximation ratio for our algorithm for Min-VC in G.

Lemma 5. *If the rounding scheme $x \mapsto y$ satisfies $y(V^*) \leqslant \frac{3}{2} \cdot x(V^*)$ and $y(V \setminus V^*) \leqslant 2 \cdot x(V \setminus V^*)$ then this gives an approximation ratio*

$$\frac{y(V)}{\mathit{OPT}} \leqslant \frac{y(V)}{x(V)} \leqslant \frac{x(V^*)}{x(V)} \cdot \frac{3}{2} + \frac{x(V \setminus V^*)}{x(V)} \cdot 2.$$

In order to apply Lemma 5 and to derive an expected approximation ratio for the algorithm, in the following we will give a lower bound on $\mathbb{E}[x(V^*)]$ and an upper bound on $x(V)$. The next lemma provides a lower bound on $x(V^*)$ in terms of the number of high-degree vertices adjacent to degree-1 and degree-2 nodes.

Lemma 6. *Let $G[V^*]$ be the subgraph of G induced by V^*. For every optimal half-integral solution x for the* Min-VC *LP, the size of the half-integral solution restricted to V^* is lower-bounded by the size of the high-degree neighborhood of degree-1 and degree-2 vertices:*

$$x(V^*) \geqslant \frac{1}{2} \cdot \big| \{u \in V | d(u) \geqslant 3 \land \exists v \in \mathcal{N}(u), d(v) \in \{1,2\}\} \big|$$

Proof. Let $V^* = X \cup Y, X = \{v \in V | d(v) \in \{1,2\}\}$ and $Y = \{u \in V | d(u) \geqslant 3 \land \exists v \in \mathcal{N}(u), d(v) \in \{1,2\}\}$. Choose some arbitrary function $f : Y \to E(X,Y)$ such that for every $u \in Y, f(u) = \{u,v\}$ for some $v \in X$ adjacent to u. $f(Y)$ consists of pairwise disjoint paths $Q_1, \dots, Q_m$ of length $\leqslant 2$, such that each path contains one or two vertices from Y. This implies $x(V^*) \geqslant m \geqslant \frac{|Y|}{2}$. ☐

First Analysis

We will now estimate the expected number of high-degree vertices adjacent to vertices of degree one or two, which – combined with the preceding Lemma 6 – gives a lower bound on $\mathbb{E}[x(V^*)]$. We prove the following theorem:

Theorem 7.

$$\begin{aligned} \mathbb{E}[x(V^*)] &\geqslant \frac{1}{2} \cdot \mathbb{E}\left[\big| \{u \in V | d(u) \geqslant 3 \land \exists v \in \mathcal{N}(u), d(v) \in \{1,2\}\} \big|\right] \\ &= \frac{1}{2} \cdot \sum_{u : d(u) \geqslant 3} \eta(u) && (1) \\ &\geqslant \frac{\mathrm{e}^{\alpha}}{2^{\beta}} \cdot \frac{\zeta(\beta) - 1 - \frac{1}{2^{\beta}}}{\zeta(\beta - 1)} && (2) \end{aligned}$$

where $\eta(u)$ is the probability that $u \in V$ has a neighbor in the set of vertices of degree one or two.

In order to provide bounds on the probability $\eta(u)$ for a vertex u of degree d of having a degree-1 or degree-2 neighbor, we consider how edges are generated in the random matching procedure of the distribution $M(\alpha, \beta)$: $d(u)$ copies of u are randomly matched with the copies of the remaining vertices $v \in V, v \neq u$. We use the following lower bound on $\eta(u)$.

Lemma 8. *For every u with $d(u) \geqslant 3$, $\eta(u) \geqslant \frac{1}{2^{\beta-1} \cdot \sum_{i=1}^{\Delta} \frac{1}{i^{\beta-1}}}$.*

Proof.

$$\begin{aligned} \eta(u) &\geqslant \Pr(\text{the first copy of } u \text{ is neighbor of a degree-2-node}) \\ &= \frac{2 \cdot \#\text{deg-2-nodes}}{\left(\sum_{v \in V} d(v)\right) - 1} \\ &\geqslant \frac{2 \cdot \frac{\mathrm{e}^{\alpha}}{2^{\beta}}}{\sum_{i=1}^{\Delta} i \cdot \frac{\mathrm{e}^{\alpha}}{i^{\beta}}} = \frac{\frac{1}{2^{\beta-1}}}{\sum_{i=1}^{\Delta} \frac{1}{i^{\beta-1}}}, \text{ where } \Delta = \mathrm{e}^{\frac{\alpha}{\beta}} \text{ is the maximum degree of } G. \end{aligned}$$

☐

In Equation 1 we substitute $\eta(u)$ by the bound given in Lemma 8 and obtain:

$$\begin{aligned}
\mathbb{E}[x(V^*)] &\geqslant \frac{1}{2} \cdot \sum_{u:d(u)\geqslant 3} \eta(u) \\
&= \frac{1}{2} \cdot \left(\sum_{i=1}^{\Delta} \frac{\mathrm{e}^{\alpha}}{i^{\beta}} - \mathrm{e}^{\alpha} - \frac{\mathrm{e}^{\alpha}}{2^{\beta}} \right) \cdot \frac{1}{2^{\beta-1} \cdot \sum_{i=1}^{\Delta} \frac{1}{i^{\beta-1}}} \\
&= \frac{\mathrm{e}^{\alpha}}{2^{\beta}} \cdot \frac{\sum_{i=1}^{\Delta} \frac{1}{i^{\beta}} - 1 - \frac{1}{2^{\beta}}}{\sum_{i=1}^{\Delta} \frac{1}{i^{\beta-1}}} \qquad (3)
\end{aligned}$$

We will now show that in Inequality 3 we can replace the terns $\sum_{i=1}^{\Delta} \frac{1}{i^{\beta}}$ and $\sum_{i=1}^{\Delta} \frac{1}{i^{\beta-1}}$ by $\zeta(\beta)$ and $\zeta(\beta-1)$, respectively. We make use of the following lemma.

Lemma 9. *For $A, B, a, b > 0$, $\frac{A}{B} \geqslant \frac{A+a}{B+b} \iff \frac{A}{B} \geqslant \frac{a}{b}$.*

Therefore, in order to show

$$\mathbb{E}[x(V^*)] \geqslant \frac{e^{\alpha}}{2^{\beta}} \cdot \frac{\zeta(\beta) - 1 - \frac{1}{2^{\beta}}}{\zeta(\beta-1)},$$

it is sufficient to show that there exists a Δ_0 such that for all $\Delta \geqslant \Delta_0$ the following holds

$$\frac{\sum_{i=1}^{\Delta} \frac{1}{i^{\beta}} - 1 - \frac{1}{2^{\beta}}}{\sum_{i=1}^{\Delta} \frac{1}{i^{\beta-1}}} \geqslant \frac{\frac{1}{(\Delta+1)^{\beta}}}{\frac{1}{(\Delta+1)^{\beta-1}}} = \frac{1}{\Delta+1}.$$

This is provided by the following lemma.

Lemma 10. *There exists a $\Delta_0 \geqslant 8$ such that for all $\Delta \geqslant \Delta_0$, $\frac{\sum_{i=1}^{\Delta} \frac{1}{i^{\beta}} - 1 - \frac{1}{2^{\beta}}}{\sum_{i=1}^{\Delta} \frac{1}{i^{\beta-1}}} \geqslant \frac{1}{\Delta+1}$.*

Proof. The above inequality is equivalent to

$$\begin{aligned}
& \sum_{i=1}^{\Delta} \frac{1}{i^{\beta}} - 1 - \frac{1}{2^{\beta}} \geqslant \sum_{i=1}^{\Delta} \frac{1}{\Delta+1} \cdot \frac{1}{i^{\beta-1}} \\
\iff \quad & \sum_{i=1}^{\Delta} \left(\frac{1}{i^{\beta}} - \frac{1}{\Delta+1} \cdot \frac{1}{i^{\beta-1}} \right) \geqslant 1 + \frac{1}{2^{\beta}} \\
\iff \quad & \sum_{i=1}^{\Delta} \frac{\Delta+1-i}{(\Delta+1)i^{\beta}} \geqslant 1 + \frac{1}{2^{\beta}} \qquad (4)
\end{aligned}$$

Suppose $\Delta \geqslant 8$, then the sum on the left-hand side of the Inequality 4 is bounded by the sum of the terms with indices $i = 1, 2, 4, 8$:

$$\begin{aligned}
\sum_{i=1}^{\Delta} \frac{\Delta+1-i}{(\Delta+1)i^{\beta}} &\geqslant \frac{\Delta}{\Delta+1} + \frac{\Delta-1}{(\Delta+1)2^{\beta}} + \frac{\Delta-3}{(\Delta+1)4^{\beta}} + \frac{\Delta-7}{(\Delta+1)8^{\beta}} \\
&= \frac{\Delta 8^{\beta} + (\Delta-1)4^{\beta} + (\Delta-3)2^{\beta} + \Delta - 7}{(\Delta+1)8^{\beta}}. \qquad (5)
\end{aligned}$$

Using Inequality 5 and the fact that $1 + \frac{1}{2^\beta} = \frac{(\Delta+1)8^\beta + (\Delta+1)4^\beta}{(\Delta+1)8^\beta}$, in order to prove Inequality 4 it is sufficient to show the following:

$$\frac{\Delta 8^\beta + (\Delta-1)4^\beta + (\Delta-3)2^\beta + \Delta - 7}{(\Delta+1)8^\beta} \overset{!}{\geqslant} \frac{(\Delta+1)8^\beta + (\Delta+1)4^\beta}{(\Delta+1)8^\beta}$$

$$\iff \qquad (\Delta-3)2^\beta + \Delta - 7 \overset{!}{\geqslant} 8^\beta + 2\cdot 4^\beta.$$

This is valid for $\Delta \geqslant \frac{8^\beta + 2\cdot 4^\beta + 6\cdot 2^\beta + 7}{1+2^\beta}$. Hence we choose $\Delta_0 = \lceil * \rceil \frac{8^\beta + 2\cdot 4^\beta + 6\cdot 2^\beta + 7}{1+2^\beta}$. □

This completes the proof of Theorem 7. The next lemma provides an upper bound for $x(V)$:

Lemma 11. $x(V) \leqslant \frac{1}{2}\zeta(\beta)\,\mathrm{e}^\alpha$

Proof. In order to get an upper bound for $x(V)$ we construct a feasible half-integral solution for G by setting $x(v) = \frac{1}{2}$ for all $v \in V$ where $\frac{1}{2}\sum_{v\in V} \leqslant \frac{1}{2}\zeta(\beta)\,\mathrm{e}^\alpha$. □

Now let us restate the main Theorem 1 and finish the proof.

Theorem. *For $\beta > 2$ the* Minimum Vertex Cover *problem in (α,β)-Power Law Graphs G can be approximated with expected approximation ratio $\rho \leqslant 2 - \frac{\zeta(\beta) - 1 - \frac{1}{2^\beta}}{2^\beta\zeta(\beta-1)\zeta(\beta)}$.*

Proof. Algorithm 1 achieves an approximation ratio of $\frac{3}{2}$ for Min-VC in the subgraph induced by V^* in G and a ratio of 2 in $G[V \setminus V^*]$, i.e.

$$\rho \leqslant \mathbb{E}\left[\frac{3}{2}\cdot\frac{x(V^*)}{x(V)} + 2\cdot\frac{x(V) - x(V^*)}{x(V)}\right] = \mathbb{E}\left[2 - \frac{1}{2}\cdot\frac{x(V^*)}{x(V)}\right].$$

From Theorem 7 and Lemma 11 we have that $\mathbb{E}[x(V^*)] \geqslant \frac{1}{2}\cdot\frac{\left(\zeta(\beta)-1-\frac{1}{2^\beta}\right)\mathrm{e}^\alpha}{2^{\beta-1}\zeta(\beta-1)}$ and $x(V) \leqslant \frac{1}{2}\cdot\zeta(\beta)\,\mathrm{e}^\alpha$. This yields

$$\mathbb{E}\left[\frac{x(V^*)}{x(V)}\right] \geqslant \frac{\frac{1}{2}\cdot\frac{\left(\zeta(\beta)-1-\frac{1}{2^\beta}\right)\mathrm{e}^\alpha}{2^{\beta-1}\zeta(\beta-1)}}{\frac{1}{2}\cdot\zeta(\beta)\,\mathrm{e}^\alpha} = \frac{\zeta(\beta) - 1 - \frac{1}{2^\beta}}{2^{\beta-1}\zeta(\beta-1)\zeta(\beta)}$$

and

$$\rho \leqslant 2 - \frac{1}{2}\cdot\frac{\zeta(\beta) - 1 - \frac{1}{2^\beta}}{2^{\beta-1}\zeta(\beta-1)\zeta(\beta)} = 2 - \frac{\zeta(\beta) - 1 - \frac{1}{2^\beta}}{2^{\beta}\zeta(\beta-1)\zeta(\beta)}$$

□

Refined Analysis for $\beta > 2.424$

We will now refine the analysis of Algorithm 1 by giving a better estimate on the probability $\eta(u,U)$ of a high-degree node u being adjacent to a vertex in the set U, i.e. a vertex of degree one or two. However, this analysis will only apply to the more restricted range of $\beta > 2.424$. Again, we will first obtain a bound on the expected approximation ratio of the algorithm in terms of the partial sums $\sum_{i=1}^{\Delta}\frac{1}{i^\beta}$ and $\sum_{i=1}^{\Delta}\frac{1}{i^{\beta-1}}$ and then show that these can be replaced by $\zeta(\beta)$ and $\zeta(\beta-1)$, respectively.

Lemma 12. *For every u with $d(u) \geqslant 3$ and $U \subseteq V$,*

$$\eta(u,U) \geqslant \frac{\sum_{i=1}^{\Delta}\frac{e^\alpha}{i^{\beta-1}} - e^{\frac{\alpha}{\beta}} + 1}{\sum_{i=1}^{\Delta}\frac{e^\alpha}{i^{\beta-1}}}\left[1 - \left(\frac{\sum_{i=1}^{\Delta}\frac{e^\alpha}{i^{\beta-1}} - d(U) - 3 + 1}{\sum_{i=1}^{\Delta}\frac{e^\alpha}{i^{\beta-1}} - 3 + 1}\right)^3\right].$$

Proof. For a given set U of vertices from G we let $d(U) = \sum_{v \in U} d(v)$. Furthermore let $\eta(u, U)$ be the probability that u is connected to at least one node in U. We obtain

$$\begin{aligned}
\eta(u,U) &= \Pr(u \text{ matches to } U) \\
&= \sum_{j=1}^{d(u)} \Pr(j\text{-th copy is first one matching to } U) \\
&= \sum_{j=1}^{d(u)} \frac{d(u)}{\sum_{i=1} \frac{e^{\alpha}}{i^{\beta-1}} - (j-1) - 1} \prod_{k=1}^{j-1} \left(1 - \frac{d(U)}{\sum_{i=1} \frac{e^{\alpha}}{i^{\beta-1}} - 1 - (k-1)}\right)
\end{aligned}$$

Now define $N = \sum_{i=1} \frac{e^{\alpha}}{i^{\beta-1}}$. We have:

$$\begin{aligned}
\eta(u,U) &= \sum_{j=1}^{d(u)} \frac{d(U)}{N-j} \prod_{k=1}^{j-1} \frac{N-d(U)-k}{N-k} \geqslant \sum_{j=1}^{d(u)} \frac{d(U)}{N-j} \left(\frac{N-d(U)-j+1}{N-j+1}\right)^{j-1} \\
&\geqslant \sum_{j=1}^{d(u)} \frac{d(U)}{N-j} \left(\frac{N-d(U)-d(u)+1}{N-d(u)+1}\right)^{j-1} \geqslant \sum_{j=1}^{d(u)} \frac{d(U)}{N} \left(\frac{N-d(U)-d(u)+1}{N-d(u)+1}\right)^{j-1} \\
&= \frac{d(U)}{N} \left[\frac{1-\left(\frac{N-d(U)-d(u)+1}{N-d(u)+1}\right)^{d(u)}}{1-\frac{N-d(U)-d(u)+1}{N-d(u)+1}}\right] \\
&= \frac{d(U)}{N} \left[1-\left(\frac{N-d(U)-d(u)+1}{N-d(u)+1}\right)^{d(u)}\right] \cdot \frac{N-d(u)+1}{d(U)} \\
&= \frac{N-d(u)+1}{N} \left[1-\left(\frac{N-d(U)-d(u)+1}{N-d(u)+1}\right)^{d(u)}\right]
\end{aligned}$$

Since the function $\left(\frac{N-d(U)-d(u)+1}{N-d(u)+1}\right)^{d(u)}$ is monotone decreasing in $d(u)$ it follows that:

$$\begin{aligned}
\eta(u,U) &\geqslant \frac{N-\Delta+1}{N} \left[1-\left(\frac{N-d(U)-3+1}{N-3+1}\right)^{3}\right] \\
&= \frac{\sum_{i=1} \frac{e^{\alpha}}{i^{\beta-1}} - e^{\frac{\alpha}{\beta}} + 1}{\sum_{i=1} \frac{e^{\alpha}}{i^{\beta-1}}} \left[1-\left(\frac{\sum_{i=1} \frac{e^{\alpha}}{i^{\beta-1}} - d(U) - 3 + 1}{\sum_{i=1} \frac{e^{\alpha}}{i^{\beta-1}} - 3 + 1}\right)^{3}\right]
\end{aligned}$$

□

Because of Equation 1 we have $\mathbb{E}[x(V^*)] \geqslant \frac{1}{2} \cdot \sum_{u:d(u)\geqslant 3} \eta(u,U)$ and we obtain the following approximation ratio:

$$\begin{aligned}\rho &\leqslant \mathbb{E}\left[2-\frac{1}{2}\cdot\frac{x(V^*)}{x(V)}\right]\\ &\leqslant 2-\frac{1}{2}\cdot\frac{\left(\sum_{i=1}^{\Delta}\frac{\mathrm{e}^\alpha}{i^\beta}-\mathrm{e}^\alpha-\frac{\mathrm{e}^\alpha}{2^\beta}\right)\cdot\frac{\sum_{i=1}^{\Delta}\frac{e^\alpha}{i^{\beta-1}}-e^{\frac{\alpha}{\beta}}+1}{\sum_{i=1}^{\Delta}\frac{e^\alpha}{i^{\beta-1}}}\left[1-\left(\frac{\sum_{i=1}^{\Delta}\frac{e^\alpha}{i^{\beta-1}}-d(U)-3+1}{\sum_{i=1}^{\Delta}\frac{e^\alpha}{i^{\beta-1}}-3+1}\right)^3\right]}{\frac{1}{2}\sum_{i=1}^{\Delta}\frac{\mathrm{e}^\alpha}{i^\beta}}\\ &=2-\underbrace{\frac{\left(\sum_{i=1}^{\Delta}\frac{1}{i^\beta}-1-\frac{1}{2^\beta}\right)\cdot\left(\sum_{i=1}^{\Delta}\frac{1}{i^{\beta-1}}-\frac{\Delta}{\mathrm{e}^\alpha}+\frac{1}{\mathrm{e}^\alpha}\right)}{\left(\sum_{i=1}^{\Delta}\frac{1}{i^{\beta-1}}\right)\cdot\left(\sum_{i=1}^{\Delta}\frac{1}{i^\beta}\right)}}_{F}\left[1-\underbrace{\left(\frac{\sum_{i=1}^{\Delta}\frac{1}{i^{\beta-1}}-\frac{d(v)}{\mathrm{e}^\alpha}-\frac{2}{\mathrm{e}^\alpha}}{\sum_{i=1}^{\Delta}\frac{1}{i^{\beta-1}}-\frac{2}{\mathrm{e}^\alpha}}\right)^3}_{C}\right]\end{aligned} \tag{6}$$

Now we show that, in Inequality 6, we can replace the partial sums $\sum_{i=1}^{\Delta}\frac{1}{i^\beta}$ and $\sum_{i=1}^{\Delta}\frac{1}{i^{\beta-1}}$ by $\zeta(\beta)$ and $\zeta(\beta-1)$ respectively. First, we consider the term C where $d(v)=\mathrm{e}^\alpha\left(1+\frac{1}{2^{\beta-1}}\right)$, i.e. the number of copies of degree-1 and degree-2 vertices:

$$C=\frac{\sum_{i=1}^{\Delta}\frac{1}{i^{\beta-1}}-\frac{\mathrm{e}^\alpha\left(1+\frac{1}{2^{\beta-1}}\right)}{\mathrm{e}^\alpha}-\frac{2}{\mathrm{e}^\alpha}}{\sum_{i=1}^{\Delta}\frac{1}{i^{\beta-1}}-\frac{2}{\mathrm{e}^\alpha}}=\frac{\sum_{i=1}^{\Delta}\frac{1}{i^{\beta-1}}-\left(1+\frac{1}{2^{\beta-1}}\right)-\frac{2}{\Delta^\beta}}{\sum_{i=1}^{\Delta}\frac{1}{i^{\beta-1}}-\frac{2}{\Delta^\beta}}$$

We show that following inequality holds true:

$$\begin{aligned} & \frac{\sum_{i=1}^{\Delta}\frac{1}{i^{\beta-1}}-\left(1+\frac{1}{2^{\beta-1}}\right)-\frac{2}{\Delta^\beta}}{\sum_{i=1}^{\Delta}\frac{1}{i^{\beta-1}}-\frac{2}{\Delta^\beta}}\leqslant\frac{\frac{1}{(\Delta+1)^{\beta-1}}+\frac{2}{\Delta^\beta}-\frac{2}{(\Delta+1)^\beta}}{\frac{1}{(\Delta+1)^{\beta-1}}+\frac{2}{\Delta^\beta}-\frac{2}{(\Delta+1)^\beta}}\\ \Longleftrightarrow\quad & \frac{\sum_{i=1}^{\Delta}\frac{1}{i^{\beta-1}}-\left(1+\frac{1}{2^{\beta-1}}\right)-\frac{2}{\Delta^\beta}}{\sum_{i=1}^{\Delta}\frac{1}{i^{\beta-1}}-\frac{2}{\Delta^\beta}}\leqslant 1\\ \Longleftrightarrow\quad & \sum_{i=1}^{\Delta}\frac{1}{i^{\beta-1}}-\left(1+\frac{1}{2^{\beta-1}}\right)-\frac{2}{\Delta^\beta}\leqslant\sum_{i=1}^{\Delta}\frac{1}{i^{\beta-1}}-\frac{2}{\Delta^\beta}\\ \Longleftrightarrow\quad & \sum_{i=1}^{\Delta}\frac{1}{i^{\beta-1}}-\left(1+\frac{1}{2^{\beta-1}}\right)\leqslant\sum_{i=1}^{\Delta}\frac{1}{i^{\beta-1}}\qquad\square\end{aligned}$$

We have

$$F=\frac{\left(\sum_{i=1}^{\Delta}\frac{1}{i^\beta}-1-\frac{1}{2^\beta}\right)\cdot\left(\sum_{i=1}^{\Delta}\frac{1}{i^{\beta-1}}-\frac{\Delta}{\mathrm{e}^\alpha}+\frac{1}{\mathrm{e}^\alpha}\right)}{\left(\sum_{i=1}^{\Delta}\frac{1}{i^{\beta-1}}\right)\cdot\left(\sum_{i=1}^{\Delta}\frac{1}{i^\beta}\right)}.$$

We let $S_\beta=\sum_{i=1}^{\Delta}\frac{1}{i^\beta}$ and $S_{\beta-1}=\sum_{i=1}^{\Delta}\frac{1}{i^{\beta-1}}$ and recall that $\mathrm{e}^\alpha=\Delta^\beta$. According to Lemma 9 it remains to show the following inequality:

$$\begin{aligned}&\frac{\left(S_\beta-1-\frac{1}{2^\beta}\right)\cdot\left(S_{\beta-1}-\frac{1}{\Delta^{\beta-1}}+\frac{1}{\Delta^\beta}\right)}{(S_{\beta-1})\cdot(S_\beta)}\\ &\geqslant\frac{1}{\frac{1}{(\Delta+1)^\beta}(S_{\beta-1})+\frac{1}{(\Delta+1)^{\beta-1}}\left(S_\beta+\frac{1}{(\Delta+1)^\beta}\right)}\cdot\left[\frac{1}{(\Delta+1)^\beta}\left(S_{\beta-1}-\frac{1}{\Delta^{\beta-1}}+\frac{1}{\Delta^\beta}\right)\right.\\ &\qquad\left.+\left(S_\beta-1-\frac{1}{2^\beta}+\frac{1}{(\Delta+1)^\beta}\right)\cdot\left(-\frac{1}{(\Delta+1)^\beta}+\frac{1}{(\Delta+1)^{\beta-1}}+\frac{1}{(\Delta)^{\beta-1}}-\frac{1}{(\Delta)^\beta}\right)\right]\end{aligned}$$

which is equivalent to

$$\begin{aligned}
&\left(S_\beta - 1 - \frac{1}{2^\beta}\right)\left(S_{\beta-1} - \frac{\Delta-1}{\Delta^\beta}\right)\left[\frac{1}{(\Delta+1)^\beta}S_{\beta-1} + \left(S_\beta + \frac{1}{(\Delta+1)^\beta}\right)\frac{1}{(\Delta+1)^{\beta-1}}\right]\\
&\geqslant \left[\frac{1}{(\Delta+1)^\beta}\left(S_{\beta-1} + \frac{\Delta-1}{\Delta^\beta}\right)\cdot S_\beta \cdot S_{\beta-1}\right.\\
&\qquad\qquad\left. + \left(S_\beta - 1 - \frac{1}{2^\beta} + \frac{1}{(\Delta+1)^\beta}\right)\cdot\left(\frac{\Delta}{(\Delta+1)^\beta} + \frac{\Delta-1}{\Delta^\beta}\right)\cdot S_\beta\cdot S_{\beta-1}\right].
\end{aligned}$$

We rearrange terms and get

$$\begin{aligned}
&S_{\beta-1}^2 S_\beta \frac{1}{(\Delta+1)^\beta} + S_\beta^2 S_{\beta-1}\left(\frac{\Delta}{(\Delta+1)^\beta} + \frac{\Delta-1}{\Delta^\beta}\right)\\
&+ S_\beta S_{\beta-1}\left[-\frac{\Delta-1}{\Delta^\beta}\cdot\frac{1}{(\Delta+1)^\beta} + \left(\frac{\Delta}{(\Delta+1)^\beta} + \frac{\Delta-1}{\Delta^\beta}\right)\left(-1 - \frac{1}{2^\beta} + \frac{1}{(\Delta+1)^\beta}\right)\right]\\
&\leqslant S_{\beta-1}^2 S_\beta \frac{1}{(\Delta+1)^\beta} + S_{\beta-1}S_\beta^2\frac{1}{(\Delta+1)^{\beta-1}}\\
&+ S_\beta S_{\beta-1}\left[\frac{1}{(\Delta+1)^\beta}\cdot\frac{1}{(\Delta+1)^{\beta-1}} - \frac{\Delta-1}{\Delta^\beta} - \left(1 + \frac{1}{2^\beta}\right)\frac{1}{(\Delta+1)^{\beta-1}}\right]\\
&+ S_\beta^2\frac{1}{(\Delta+1)^{\beta-1}} + S_{\beta-1}^2\left(-1 - \frac{1}{2^\beta}\right)\frac{1}{(\Delta+1)^\beta}\\
&+ S_{\beta-1}\left[-\left(1 + \frac{1}{2^\beta}\right)\frac{1}{(\Delta+1)^\beta}\cdot\frac{1}{(\Delta+1)^{\beta-1}}\left(1 + \frac{1}{2^\beta}\right)\frac{\Delta-1}{\Delta^\beta}\cdot\frac{1}{(\Delta+1)^\beta}\right]\\
&+ S_\beta\left[\left(-1 - \frac{1}{2^\beta}\right)\frac{1-\Delta}{\Delta^\beta}\cdot\frac{1}{(\Delta+1)^{\beta-1}} - \frac{\Delta-1}{\Delta^\beta}\cdot\frac{1}{(\Delta+1)^\beta}\cdot\frac{1}{(\Delta+1)^{\beta-1}}\right]\\
&\qquad\qquad\qquad + \left(1 + \frac{1}{2^\beta}\right)\frac{\Delta-1}{\Delta^\beta}\cdot\frac{1}{(\Delta+1)^\beta}\cdot\frac{1}{(\Delta+1)^{\beta-1}}
\end{aligned}\tag{7}$$

The following lemma shows that in order to prove Inequality 7 it is sufficient to show the respective inequality given by the terms of slowest convergence as Δ goes to infinity.

Lemma 13. *Let $f_\beta, g_\beta, F_\beta, G_\beta$ be functions of Δ depending on the parameter β with $|g_\beta|, |G_\beta| \leqslant c$ for a constant c depending only on β. Then $f_\beta(\Delta) < F_\beta(\Delta)$ for almost all Δ implies*

$$f_\beta(\Delta)\cdot\frac{1}{\Delta^{\beta-1}} + g_\beta(\Delta)\cdot\frac{1}{\Delta^\beta} \leqslant F_\beta(\Delta)\cdot\frac{1}{\Delta^{\beta-1}} + G_\beta(\Delta)\cdot\frac{1}{\Delta^\beta}$$

for all but finitely many Δ.

Hence it remains to show that

$$\begin{aligned}
&S_\beta^2 S_{\beta-1}\left(\frac{\Delta}{(\Delta+1)^\beta} + \frac{\Delta-1}{\Delta^\beta}\right) - S_\beta S_{\beta-1}\left(1 + \frac{1}{2^\beta}\right)\left(\frac{\Delta}{(\Delta+1)^\beta} + \frac{\Delta-1}{\Delta^\beta}\right)\\
&\qquad\qquad\qquad\qquad \leqslant S_\beta S_{\beta-1}(-1)\cdot\frac{\Delta-1}{\Delta^\beta} + S_{\beta-1}S_\beta^2\frac{1}{(\Delta+1)^{\beta-1}}
\end{aligned}$$

which holds true if and only if

$$\begin{aligned}
&S_\beta^2 S_{\beta-1}\left(\frac{\Delta}{(\Delta+1)^\beta} + \frac{\Delta-1}{\Delta^\beta}\right)\\
&\qquad \leqslant S_\beta S_{\beta-1}\left[\left(1 + \frac{1}{2^\beta}\right)\left(\frac{\Delta}{(\Delta+1)^\beta} + \frac{\Delta-1}{\Delta^\beta}\right) - \frac{\Delta-1}{\Delta^\beta}\right] + S_{\beta-1}S_\beta^2\frac{1}{(\Delta+1)^{\beta-1}}
\end{aligned}$$

which can be rewritten as

$$\begin{aligned}
& S_\beta^2 S_{\beta-1}\left(\frac{1}{(\Delta+1)^\beta}+\frac{\Delta-1}{\Delta^\beta}\right) \leqslant S_\beta S_{\beta-1}\left[\left(1+\frac{1}{2^\beta}\right)\frac{\Delta}{(\Delta+1)^\beta}+\frac{1}{2^\beta}\cdot\frac{\Delta-1}{\Delta^\beta}\right] \\
\Longleftrightarrow\quad & S_\beta\left(\frac{\Delta-1}{\Delta^\beta}\right) \leqslant \left(1+\frac{1}{2^\beta}\right)\frac{\Delta}{(\Delta+1)^\beta}+\frac{1}{2^\beta}\cdot\frac{\Delta-1}{\Delta^\beta} \\
\Longleftrightarrow\quad & S_\beta(\Delta-1) \leqslant \left(1+\frac{1}{2^\beta}\right)\frac{\Delta^{\beta+1}}{(\Delta+1)^\beta}+\frac{1}{2^\beta}\cdot(\Delta-1) \\
\Longleftrightarrow\quad & \left(S_\beta-\frac{1}{2^\beta}\right)(\Delta-1) \leqslant \left(1+\frac{1}{2^\beta}\right)\frac{\Delta^{\beta+1}}{(\Delta+1)^\beta} \qquad (8)
\end{aligned}$$

Now Inequality 8 follows from the observation that for all $\beta > 2.424$, $S_\beta - \frac{1}{2^\beta} < 1 + \frac{1}{2^\beta}$.

Finally we have shown the following theorem.

Theorem. *For all $\beta > 2.424$ the* Minimum Vertex Cover *problem on (α,β)-Power Law Graphs G can be approximated with expected approximation ratio*

$$\rho \leqslant 2 - \frac{\left(\zeta(\beta)-1-\frac{1}{2^\beta}\right)\cdot\left(\zeta(\beta-1)-\frac{\Delta}{e^\alpha}+\frac{1}{e^\alpha}\right)}{\zeta(\beta-1)\cdot\zeta(\beta)}\left[1-\left(\frac{\zeta(\beta-1)-\left(1+\frac{1}{2^{\beta-1}}\right)-\frac{2}{e^\alpha}}{\zeta(\beta-1)-\frac{2}{e^\alpha}}\right)^3\right]$$

This converges to

$$\rho \leqslant 2 - \frac{\left(\zeta(\beta)-1-\frac{1}{2^\beta}\right)\cdot\zeta(\beta-1)}{\zeta(\beta-1)\cdot\zeta(\beta)}\left[1-\left(\frac{\zeta(\beta-1)-\left(1+\frac{1}{2^{\beta-1}}\right)}{\zeta(\beta-1)}\right)^3\right]$$

as $\alpha \to \infty$.

4 Conclusion

In Section 3 we presented a new approximation algorithm for Min-VC in (α,β)-PLG with expected approximation ratio of $\rho \leqslant 2-\frac{\zeta(\beta)-1-\frac{1}{2^\beta}}{2^\beta\zeta(\beta-1)\zeta(\beta)}$ in our first analysis of Section 3.2. Moreover, in our refined analysis we showed for $\beta > 2.424$ an expected asymptotic approximation ratio of $\rho' \leqslant 2-\frac{\left(\zeta(\beta)-1-\frac{1}{2^\beta}\right)\cdot\zeta(\beta-1)}{\zeta(\beta-1)\cdot\zeta(\beta)}\left[1-\left(\frac{\zeta(\beta-1)-\left(1+\frac{1}{2^{\beta-1}}\right)}{\zeta(\beta-1)}\right)^3\right]$. The algorithm itself basically consists of a deterministic rounding procedure on a half-integral solution for Min-VC (c.f. Algorithm 1). We showed that this rounding procedure yields an approximation ratio of $\frac{3}{2}$ in the subgraph induced by the low-degree vertices of the (α,β)-PLG and a 2-approximation in the residual graph.

Further research will be directed towards extending the improved analysis also to the range $\beta < 2.424$ and towards investigating the approximability of Min-VC in other PLG-Models, e.g. the Preferential Attachment Model in [BA99].

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