Approximate Counting of Matchings in Sparse Hypergraphs

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Abstract. In this paper we give a fully polynomial randomized approximation scheme (FPRAS) for the number of all matchings in hypergraphs belonging to a class of sparse, uniform hypergraphs. Our method is based on a generalization of the canonical path method to the case of uniform hypergraphs.

1 Introduction

A hypergraph $H = (V, E)$ is a finite set of vertices V together with a family E of distinct, nonempty subsets of vertices called edges. In this paper we consider *k-uniform hypergraphs* (called also *k-graphs*) in which, for a fixed $k \geq 2$, each edge is of size k. *A matching* in a hypergraph is a set of disjoint edges. We will often identify a matching M with the hypergraph $H[M]$ induced by M in H. *The intersection graph* of a hypergraph H is a graph $L := L(H)$ with vertex set $V(L) = E(H)$ and the edges set $E(L)$ consisting of all intersecting pairs of edges of H. When H is a graph, the intersection graph $L(H)$ is called *the line graph* of H.

In a seminal paper [11], Jerrum and Sinclair constructed an FPRAS (see Section 1.3 for the definition) for counting the number of all matchings in graphs (a.k.a. the monomer-dimer problem) based on an ingenious technique of canonical paths. The method was extended later in [12] to solve the permanent problem. Here we test for what classes of hypergraphs their method can be carried over. It turns out that, given a k-graph H, we can adopt the proof for graphs only when for every two matchings M, M' in H the intersection graph $L = L(M \cup M')$ between M and M' has maximum degree $\Delta(L) \leq 2$. This happens if and only if H contains no *3-comb*, i.e. a k-graph H_0 consisting of a matching $\{e_1, e_2, e_3\}$ and one extra edge e_4 such that $|e_4 \cap e_i| \geq 1$ for $i = 1, 2, 3$ (see Fig.1(a)). Let us denote by $\mathcal{H}_0$ the family of all k-graphs which do not contain H_0. Our main result is the following hypergraph generalization of the Jerrum-Sinclair theorem. In fact, they considered the weighted case, while we, for clarity, assume that the

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hypergraphs are unweighted. However, the weighted case can be handled in a similar manner.

Theorem 1. *There exists an FPRAS for the problem of counting all matchings in k-graphs $H \in \mathcal{H}_0$.*

The requirement that $H \in \mathcal{H}_0$ is quite restrictive and causes the hypergraph to be rather sparse. We can characterize family $\mathcal{H}_0$ in terms of the intersection graph $L(H)$. Namely, a k-graph $H \in \mathcal{H}_0$ if and only if the intersection graph $L(H)$ of H is *claw-free*, that is $L(H)$ does not contain an induced subgraph isomorphic to the star $K_{1,3}$.

Theorem 1 is proved in Section 2. In section 3 we give several examples of classes of k-graphs which belong to $\mathcal{H}_0$. Finally, in Section 4 we discuss obstacles preventing us from extending our result to all hypergraphs.

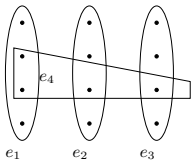
1.1 Motivation from statistical physics

In 1972 Heilmann and Lieb [9] studied monomer-dimer systems, which in the graph theoretic language correspond to (weighted) matchings in graphs. In physical applications these graphs are typically some (infinite) regular lattices. Dimers represent diatomic molecules which occupy disjoint pairs of adjacent vertices of the lattice and monomers are the remaining vertices. Heilmann and Lieb proved that the associated Gibbs measure is unique (in other words, there is no phase transition). They did it by proving that the roots of the generating matching polynomial of any graph are all real, equivalently that the roots of the hard core partition function (independence polynomial) of any line graph are all real. The latter result was later extended to all claw-free graphs by Chudnovsky and Seymour [6]. The uniqueness of Gibbs measure on d-dimensional latticed was reproved in a slightly stronger form and by a completely different method by Van der Berg [3].

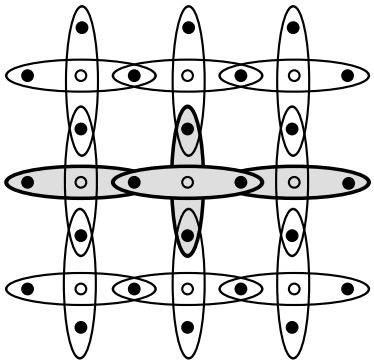
Hypergraphs may be at hand when instead of diatomic molecules bigger molecules (polymers) are considered which, again, can occupy "adjacent", disjoint sets of vertices of a lattice. As long as the hypergraph lattice H belongs to the family $\mathcal{H}_0$, the intersection graph $L(H)$ is claw-free (because H contains no 3-comb) and, by the result of [6] combined with the proof from [9] there is no phase transition either. However, it is possible to have a phase transition for a monomer-trimer system (cf. [8]). Interestingly, the example given by Heilmann (the decorated, or subdivided, square lattice with hyperedges corresponding to the collinear triples with midpoints at the branching points of the original square lattice) is a 3-uniform hypergraph containing 3-combs, and thus its intersection graph is *not* claw-free (see Fig.1(b)).

1.2 Related results

Recently, an alternative approach to constructing counting schemes for graphs has been developed based on the concept of spatial correlation decay. This resulted in deterministic fully polynomial time approximation schemes (FPTAS)



(a) a 4-uniform 3-comb



(b) Heilmann's 3-graph lattice; the shaded edges form a 3-comb

Fig. 1.

for counting independent sets in graphs with degree at most five ([15]), proper colorings ([7]), and matchings in graphs of bounded degree ([2]). It is not clear if these methods can be extended to hypergraphs.

The above mentioned FPTAS for counting independent sets in graphs implies an FPTAS for counting matchings in hypergraphs whose intersection graphs have maximum degree at most five. This is the case of the hexagon based lattice (see Subsection 3.1) and, with high probability, of the random hypergraphs discussed in Subsections 3.2 and 3.3. More importantly, this is the case also of the the Heilmann lattice described in the previous subsection (the maximum degree of its intersection graph is three), which undermines our temptation to link the absence of phase transition for a hypergraph lattice with the absence of a 3-comb, that is with the claw-freeness of the intersection graph of the lattice.

As far as hypergraphs are concerned, the authors of [4] showed that, under certain conditions, the Glauber dynamics for independent sets in a hypergraph, as well as the Glauber dynamics for proper q-colorings of a hypergraph mix rapidly. It is doubtful, however, if the path coupling technique applied there can be of any use for the problem of counting matchings in hypergraphs. Nevertheless, paper [4] marks a new line of research, as there have been no results on approximate counting in hypergraphs before. The only other paper devoted to counting matchings in hypergraphs we are aware of is [1], where Barvinok and Samorodnitsky compute the partition function for matchings in hypergraphs under some restrictions on the weights of edges. In particular they are able to distinguish in polynomial time between hypergraphs that have sufficiently many perfect matchings and hypergraphs that do not have nearly perfect matchings.

1.3 Approximate Counting and Unform Sampling

Given $\epsilon > 0$ and $\delta > 0$, we say that a random variable Y is an (ϵ, δ)-*approximation* of a constant C if

$$\mathrm{P}\left(|Y - C| \geq \epsilon C\right) \leq \delta.$$

Consider a problem of evaluating a function f over a set of input strings Σ^*.

Definition 2. A randomized algorithm is called a *fully polynomial randomized approximation scheme (FPRAS) for* f if for every triple (ϵ, δ, x) with $\epsilon > 0,\ \delta > 0$, and $x \in \Sigma^*$, the algorithm returns an (ϵ, δ)-*approximation* Y of $f(x)$ and runs in time polynomial in $1/\epsilon$, $\log(1/\delta)$, and $|x|$.

Consider a counting problem, that is, a problem of computing $f(x) = |\Omega(x)|$, where $\Omega(x)$ is a well defined finite set associated with x (think of the set of all matchings in a hypergraph). As it turns out (see below), to construct an FPRAS for such a problem it is sufficient to be able to efficiently sample an element of $\Omega(x)$ almost uniformly at random. To make it precise, given $\epsilon > 0$, we say that a probability distribution $\mathrm{P} : 2^{\Omega} \to [0, 1]$ over a finite sample space Ω is ϵ-uniform if for every $S \subseteq \Omega$

$$\left|\mathrm{P}(S) - \frac{|S|}{|\Omega|}\right| \leq \epsilon,$$

that is, if the total variation distance, $d_{TV}(\mathrm{P}, \frac{1}{|\Omega|})$, between the two distributions is bounded by ϵ.

Definition 3. A randomized algorithm is called a *fully polynomial almost uniform sampler (FPAUS)* for a counting problem $|\Omega(x)|$ if for every pair (ϵ, x) with $\epsilon > 0$ and $x \in \Sigma^*$, the algorithm samples $\omega \in \Omega$ according to an ϵ-uniform distribution P and runs in time polynomial in $1/\epsilon$ and $|x|$.

It has been proved by Jerrum, Valiant, and Vazirani [13] that for a broad class of counting problems, called self-reducible, including the matching problem, knowing an FPAUS allows one to construct an FPRAS. For a proof in the graph case see Proposition 3.4 in [10]. The hypergraph case follows mutatis mutandis. Thus, the proof of Theorem 1 reduces to constructing an FPAUS for matchings in H.

In fact, this approach has been invented for matchings in graphs already by Broder in [5], and successfully executed by Jerrum and Sinclair in [11]. In their version the main steps of finding an efficient FPAUS for matchings in a graph H were

- a construction of an ergodic time-reversible, symmetric Markov chain $\mathcal{MC}(H)$ whose state space Ω consists of all matchings in H;
- a proof that $\mathcal{MC}(H)$ is rapidly mixing.

1.4 Rapid mixing

Given an arbitrary probability distribution P_0 on the state space Ω, let us define the mixing time $t_{mix}(\epsilon)$ of a Markov chain $\mathcal{MC}$ as

$$t_{mix}(\epsilon) = \min\{t : d_{TV}(\mathrm{P}_t, \tfrac{1}{|\Omega|}) \leq \epsilon\},$$

where P_t is the chain's state distribution after t steps, beginning from the initial distribution P_0. Recall that if an ergodic time-reversible Markov chain is symmetric, i.e., the transition probabilities satisfy $p_{ij} = p_{ji}$ for all $i, j \in \Omega$, then its unique stationary distribution is uniform (cf. [10]). In that case we define the transition graph $G_{\mathcal{MC}} = G$ of $\mathcal{MC}$ as a graph on the vertex set $V(G) = \Omega$ and the edge set $E(G) = \{\{i, j\} : p_{ij} > 0\}$. Note that G is undirected but, possibly, with loops. The pivotal role in estimating the rate of convergence of $\mathcal{MC}$ to its uniform stationary distribution is played by an expansion parameter, called *the conductance* and denoted $\Phi(\mathcal{MC})$ which in the symmetric case is defined by a simplified formula

$$\Phi := \Phi(\mathcal{MC}) = min_{S\subseteq\Omega:\, 0<|S|\leq\frac{1}{2}|\Omega|} \frac{\sum_{ij\in G} p_{ij}}{|S|}. \tag{1}$$

Indeed, it follows from Theorem 2.2 in [11] that if $p_{ii} \geq \frac{1}{2}$ for all $i \in \Omega$ then

$$d_{TV}(\mathrm{P}_t, \tfrac{1}{|\Omega|}) \leq |\Omega|^2 \left(1 - \Phi^2/2\right)^t, \tag{2}$$

regardless of the initial distribution P_0, and consequently,

$$t_{mix}(\epsilon) \leq \frac{2}{\Phi^2}\left(2\log|\Omega| + \log\epsilon^{-1}\right) \tag{3}$$

Hence, it becomes crucial to estimate the conductance from below by the reciprocal of a polynomial in the input size. To this end, observe that

$$\Phi(\mathcal{MC}) \geq min_{S\subseteq\Omega:\, 0<|S|\leq\frac{1}{2}|\Omega|} \frac{p_{\min}|cut(S)|}{|S|}, \tag{4}$$

where $cut(S)$ is the edge-cut of G defined by S, and

$$p_{\min} = \min\{p_{ij} :\ \{i, j\} \in G,\ i \neq j\}.$$

For Markov chains on matchings of an n-vertex k-graph H, denoted further by $\mathcal{MC}(H)$, to bound $|cut(S)|$, Jerrum and Sinclair introduced their method of canonical paths which boils down to:

- defining a *canonical path* in G for every pair of matchings (I, F) in H;
- bounding from above the number of canonical paths containing a prescribed transition (an edge of G) by $poly(n)|\Omega|$.

Since every canonical path between a matching in S and a matching in the complement of S must go through an edge of $cut(S)$, we have, clearly,

$$|cut(S)| \geq \frac{|S|(|\Omega| - |S|)}{poly(n)|\Omega|} \geq \frac{|S|}{2poly(n)} \tag{5}$$

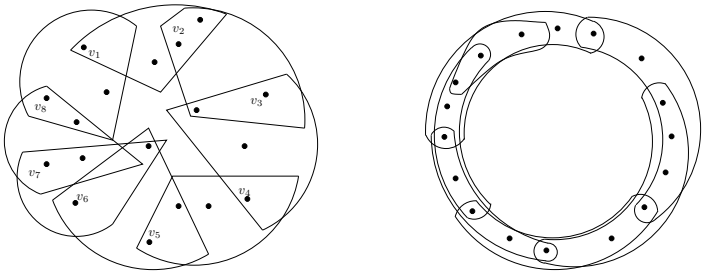
and, consequently,

$$\Phi(\mathcal{MC}(H)) \geq \frac{p_{\min}}{poly(n)}. \tag{6}$$

Our plan for proving Theorem 1 is to basically follow the footprints of [11] for as long as it is feasible for the more complex structure of hypergraphs. Our proof is given in the next section, while Section 3 contains examples of k-uniform hypergraphs to which Theorem 1 applies. But first we collect together various definitions of cycles in hypergraphs, a couple of which will be used later in the paper.

1.5 Cycles in hypergraphs

A hypergraph with edges $\{e_1, \ldots, e_m\}$, where $m \geq 3$, is *a Berge-cycle* if there is *a subset* $\{v_1, \ldots, v_m\}$ of its vertex set (called *the core*) such that for every $i = 1, \ldots, m$, the vertices v_i and v_{i+1} belong to e_i where $v_{m+1} := v_1$. (See Fig.2(a) for an example of a Berge-cycle). The structure of a Berge-cycle, in general, may be quite chaotic, not resembling what we think a cycle should look like.



(a) a 4-uniform Berge cycle (b) a necklace

Fig. 2.

A necklace is a Berge-cycle whose *all* vertices can be ordered cyclically in such a way that every edge forms a segment of this ordering – see Fig.2(b). (Then the first vertices of each edge form the core.) A necklace C is called *loose* if $\Delta(C) \leq 2$. *An open loose necklace* is a hypergraph with edge set $\{e_1, \ldots, e_m\}$, where $m \geq 1$, such that for every $1 \leq i < j \leq m$, $e_i \cap e_j \neq \emptyset$ if and only if $j = i + 1$.

Moving to k-uniform hypergraphs, for a pair of natural numbers $k \geq 3$ and $1 \leq \ell \leq k-1$, define an *ℓ-overlapping k-cycle* $C_n^{(k)}(\ell)$ as a necklace of n vertices in which every two consecutive edges (in the natural ordering of the edges induced by the ordering of the vertices) share exactly ℓ vertices. The case of $\ell = k-1$ is referred to as the *tight* cycle. Note that $C_n^{(k)}(\ell)$ has precisely $\frac{n}{k-\ell}$ edges. Note also that for $\ell \leq k/2$ the cycle $C_n^{(k)}(\ell)$ is a loose necklace.

2 The proof of Theorem 1

In this section we define a Markov chain whose states are the matchings of a k-uniform hypergraph H and then prove Theorem 1 by showing that the chain is rapidly mixing to a uniform stationary distribution.

2.1 The Markov Chain

Given a k-graph $H = (V, E)$, $|V| = n$, let $\Omega(H)$ denote the set of all matchings in H. We define a Markov chain $\mathcal{MC}(H) = (X_t)_{t=0}^{\infty}$ with state space $\Omega(H)$ as follows. Set $X_0 = \emptyset$ and for $t \geq 0$, let X_t be a matching $M = \{h_1, h_2, \ldots, h_s\}$ with $h_i \in H$, $0 \leq s \leq n/k$. Choose an edge $h \in H$ uniformly at random and consider the set $S_h := \{i : h \cap h_i \neq \emptyset,\ i = 1, \ldots, s\}$. The following transitions from X_t are allowed in $\mathcal{MC}(H)$:

(-) if $h \in M$ then $M' := M - h$,
(+) if $h \notin M$ and $|S_h| = 0$ then $M' := M + h$,
(+/-) if $h \notin M$ and $S_h = \{j\}$ then $M' := M + h - h_j$,
(0) if $h \notin M$ and $|S_h| \geq 2$ then $M' := M$.

Finally, with probability $1/2$ set $X_{t+1} := M'$, else $X_{t+1} := X_t$.

Fact 4. *The Markov chain $\mathcal{MC}(H)$ is ergodic and symmetric.*

Proof. First note that this chain is irreducible (one can get from any matching to any other matching by a sequence of above transitions) and aperiodic (due to loops), and so it is ergodic. To prove the symmetry of $\mathcal{MC}(H)$, note that for two different matchings $M, M' \in \Omega(H)$, the transition probability

$$\mathrm{P}_{M,M'} = \begin{cases} \frac{1}{|H|} & \text{if } |M \oplus M'| = 1 \\ \frac{1}{|H|} & \text{if } M \oplus M' = \{e, f\}, e \cap f \neq \emptyset \\ 0 & \text{otherwise .} \end{cases} \tag{7}$$

Thus, $\mathrm{P}_{M,M'} = \mathrm{P}_{M',M}$. □

The above fact implies that $\mathcal{MC}(H)$ converges to a stationary distribution that is uniform over $\Omega(H)$. Moreover, it follows from equation (7) that

$$p_{\min} = \min\{P_{M,M'} : \{M, M'\} \in G,\ M \neq M'\} = \frac{1}{|H|} \geq n^{-k}. \tag{8}$$

2.2 Canonical Paths

In this section we will define canonical paths, a tool used for estimating the mixing time of the Markov chain $\mathcal{MC}(H)$ introduced in the previous subsection. For brevity, loose necklaces will be called *cycles*, while open loose necklaces will be called *paths*. (Note that a pair of edges sharing at least two vertices is a path, not z cycle.)

Set $V(H) = \{1, 2, \ldots, n\}$ and $\min S = \min\{i : i \in S\}$ for any $S \subseteq V(H)$. Let (I, F) be an ordered pair of matchings in $\Omega(H)$ (we might think of them as the initial and the final matching of the canonical path-to-be). The symmetric difference $I \oplus F$ is a hypergraph with $\Delta(I \oplus F) \leq 2$ and, due to the assumption that $H \in \mathcal{H}_0$, also $\Delta(L(I \oplus F)) \leq 2$, that is, in $I \oplus F$ every edge intersects at most two other edges. Hence, each component of $I \oplus F$ is a path or a cycle, in which the edges of I alternate with the edges F. In particular, each cycle-component has an even number of edges.

Let us order the components $Q_1, \ldots, Q_q$ of $I \oplus F$ so that $\min V(Q_1) < \cdots < \min V(Q_q)$. We construct the canonical path $\gamma(I, F) = (M_0, \ldots, M_t)$ in the transition graph G by setting $M_0 = I$ and then modifying the current matching by transitions (+), (-), or (+/-), while traversing the components $Q_1, \ldots, Q_q$ as follows. For the sake of uniqueness of the canonical path, each component will be traversed from a well defined starting point (an edge e_1) and in a well defined direction $e_1, e_2, \ldots e_s$. Of, course, for a path there are just two starting points (which determine directions), while for a cycle there are s starting points and two directions from each. The particular rules for choosing the starting point and direction are quite arbitrary and do not really matter for us. Suppose that we have already constructed matchings $M_0, M_1, \ldots, M_j$ and traversed so far the components $Q_1, \ldots, Q_{r-1}$.

If Q_r is an even path then we assume that $e_1 \in F$ (and so $e_s \in I$) and take $M_{j+1} = M_j + e_1 - e_2$, $M_{j+2} = M_{j+1} + e_3 - e_4$,..., $M_{j+s/2} = M_{j+s/2-1} + e_{s-1} - e_s$. If Q_r is an odd path then we assume that $\min(e_1 \cap e_2) < \min(e_{s-1} \cap e_s)$. If $e_1, e_s \in I$ then take $M_{j+1} = M_j - e_1$, $M_{j+2} = M_{j+1} + e_2 - e_3$, $M_{j+3} = M_{j+2} + e_4 - e_5$, ..., $M_{j+(s+1)/2} = M_{j+(s-1)/2} + e_{s-1} - e_s$. If $e_1, e_s \in F$, we apply the sequence of transitions $M_{j+1} = M_j + e_1 - e_2$, $M_{j+2} = M_{j+1} + e_3 - e_4$,...,$M_{j+(s-1)/2} = M_{j+(s-3)/2} + e_{s-2} - e_{s-1}$, and $M_{j+(s+1)/2} = M_{j+(s-1)/2} + e_s$. Finally, if $Q_r = (e_1, \ldots, e_s)$ is a cycle then we assume that $\min e_1 = \min(V(Q_r) \cap V(I))$ and $\min(e_2 \cap e_3) > \min(e_{s-1} \cap e_s)$, and follow the sequence of transitions $M_{j+1} = M_j - e_1$, $M_{j+2} = M_{j+1} + e_2 - e_3$, $M_{j+3} = M_{j+2} + e_4 - e_5$, ...,$M_{j+s/2} = M_{j+s/2-1} + e_{s-2} - e_{s-1}$, and $M_{j+s/2+1} = M_{j+s/2} + e_s$.

We call the component Q_r of $I \oplus F$ *the venue* of the transition (M_j, M_{j+1}) (on the canonical path $\gamma(I, F)$) if $M_j \oplus M_{j+1} \subseteq E(Q_r)$. Note that the obtained sequence $\gamma(I, F) = (M_0, \ldots, M_t)$ is unique and satisfies the following properties:

(a) $M_0 = I$ and $M_t = F$,
(b) for every $j = 0, \ldots, t-1$, the pair $\{M_j, M_{j+1}\}$ is an edge of the transition graph $G_{\mathcal{MC}(H)}$,
(c) for every $j = 0, \ldots, t$, we have $I \cap F \subseteq M_j \subseteq I \cup F$,

(d) for every $j = 0, \ldots, t$, we have $F \cap \bigcup_{i=1}^{r-1} Q_i \subseteq M_j$ and $I \cap \bigcup_{i=r+1}^{q} Q_i \subseteq M_j$, where Q_r is the venue of (M_j, M_{j+1}).

2.3 Bounding the cuts

Fix a transition edge (M, M') in $G_{\mathcal{MC}(H)}$. Let $\Pi_{M,M'} = \{(I, F) : (M, M') \in \gamma(I, F)\}$ be the set of canonical paths containing the transition edge (M, M'). Our goal is to show that

$$|\Pi_{M,M'}| \leq |\Omega'(H)|, \tag{9}$$

where

$$\Omega'(H) = \{H' \subseteq H : \exists e \in H' \text{ such that } H' - e \in \Omega(H)\}.$$

Note that $|\Omega'(H)| \leq |\{(M, e) : M \in \Omega(H),\ e \in H\}| \leq n^k |\Omega(H)|$ and $\log |\Omega(H)| = O(n \log n)$. Thus, in view of the remarks at the end of Section 1, the estimates (3), (5), (6), (8), and (9) yield a polynomial bound on $t_{mix}(\epsilon)$ and thus complete the proof of Theorem 1.

We will prove (9) by defining a function $\eta_{M,M'} : \Pi_{M,M'} \to \Omega'(H)$ and showing that $\eta_{M,M'}$ is an injection. Fix $(I, F) \in \Pi_{M,M'}$ and define

$$\eta_{M,M'}(I, F) = (I \oplus F) \oplus (M \cup M'). \tag{10}$$

Fact 5. *For all $I, F \in \Pi_{M,M'}$ we have $\eta_{M.M'}(I, F) \in \Omega'$.*

Proof. If $(I, F) \in \Pi_{M,M'}$ then the canonical path $\gamma(I, F) = (M_0, \ldots, M_t)$ contains a consecutive pair $M_j = M$ and $M_{j+1} = M'$ for some $j \in \{0, \ldots, t\}$. Let Q_r be the component of $I \oplus F$ which is the venue of (M, M') on $\gamma(I, F)$. By the construction of $\gamma(I, F)$ it follows that $\eta_{M,M'}$ is a matching, unless Q_r is a cycle $(e_1, \ldots, e_s)$ and $M' = M + e_\ell - e_{\ell+1}$ for some $\ell \in \{2, 4, \ldots, s-2\}$. But then, by property (d) above, we have

$$\eta_{M,M'} = I \cap \bigcup_{i=1}^{r-1} Q_i \cup F \bigcup_{i=r+1}^{q} Q_i \cup \{e_1, e_3, \ldots, e_{\ell-1}, e_{\ell+2}, \ldots, e_s\}.$$

Hence, $\eta_{M,M'} - e_1 \in \Omega(H)$, and, consequently, $\eta_{M.M'} \in \Omega'(H)$. □

Fact 6. *The mapping $\eta_{M,M'} : \Pi_{M,M'} \to \Omega'(H)$ is injective.*

Proof. We will prove this fact by showing that any value η of this function uniquely determines the pair (I, F) for which $\eta_{M,M'}(I, F) = \eta$. Given $\eta_{M,M'}(I, F)$ we can recover $I \oplus F$ by reversing equation (10):

$$I \oplus F = \eta \oplus (M \cup M').$$

By property (c), we immediately have $I \cap F = M \setminus (I \oplus F)$. It remains to distinguish between the edges of $I \oplus F$ which belong to I and to F. First observe that we can recover the original ordering of the components $Q_1, \ldots, Q_q$ of $I \oplus F$ (by computing $\min V(Q_i)$ for all i), as well as the venue Q_r of the transition

(M, M') on the canonical path $\gamma(I, F)$ (by locating $M \oplus M'$). By property (d), for every $i < r$ we have $Q_i \cap M \subseteq F$, while for every $i > r$ we have $Q_i \cap M \subseteq I$. To reconstruct I and F on Q_r, note that it suffices to identify just one edge of Q_r and then follow the alternating pattern of I and F on Q_r. To this end, note that $|M \setminus M'| \leq 1$ and $|M' \setminus M| \leq 1$ but $|M \oplus M'| \geq 1$. If $M \setminus M' = \{e\}$ then $e \in I$. If $M \setminus M' = \emptyset$ then the unique edge which belongs to $M' \setminus M$ is in F. □

3 Hypergraphs with no 3-combs

In this section we give a couple of examples of classes of k-uniform hypergraphs which belong to family $\mathcal{H}_0$. A hypergraph is called *simple* (a.k.a. linear) when no two edges share two vertices, that is, the maximum pair degree is one. The proofs of the facts stated in this section will be given in the journal version of the paper.

3.1 Hypergraphs based on hexagonal lattices

The kagome lattice is the line graph of the hexagonal lattice. We construct a 3-graph H by replacing each edge $e = uv$ of the hexagonal lattice G with a triple uw_ev, where w_e, $e \in G$, are new and distinct vertices. Then, the intersection graph $L(H)$ is still the kagome lattice and thus, H contains no 3-comb. Note, however, that $\Delta(L(H)) = 4$ and, in view of Weitz's result in [15], our Theorem 1 is not new for such H.

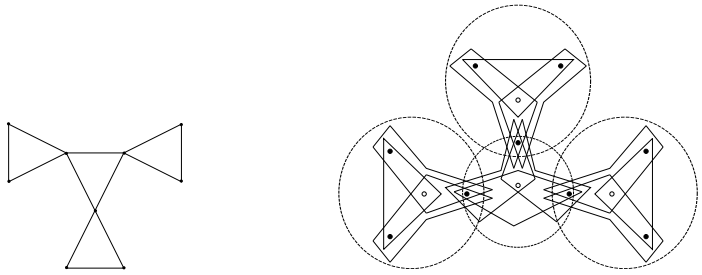
3.2 Random hypergraphs

Consider a random binomial k-graph $H = H^{(k)}(n, p)$ where each k-tuple of vertices becomes an edge, independently, with probability $p = p(n)$. To ensure the absence of copies of the 3-comb H_0 in H we need $p = o(n^{-k+3/4})$. Indeed, then the expected number of 3-combs is $O(n^{4k-3}p^4) = o(1)$. Consequently, almost all k-graphs on n vertices and with $m = o(n^{3/4})$ edges are H_0-free. Such k-graphs, however, are very sparse. For instance, typically the maximum vertex degree is three and the maximum pair degree is one, that is, they are simple.

3.3 3-graphs on triangles of random graphs

A *windmill* is a graph consisting of four triangles: one central triangle and three other, mutually disjoint triangles, each of which shares one vertex with the central triangle (see Fig.3(a)). For a graph G define the *triangle 3-graph* $H = T(G)$ where the hyperedges correspond to the vertex sets of the triangles of G. Clearly, if G is windmill-free then $H \in \mathcal{H}_0$. The matchings of H correspond to K_3-matchings of G, i.e. to vertex-disjoint unions of triangles in G.

Note that almost all graphs with n vertices and $m = o(n^{5/4})$ edges are windmill-free. Indeed, a random graph $G(n, p)$, $p = o(n^{-3/4})$, a.a.s. has no windmills because the expected number of windmills is $O(n^9p^{12}) = o(1)$. To have a



(a) a windmill (b) a subdivided 3-graph

Fig. 3.

better insight into the structure of such graphs, observe that for such p, a typical $G(n,p)$ has $o(n^{5/4})$ edges, $o(n^{3/4})$ triangles and $o(n^4p^5) = o(n^{1/4})$ pairs of triangles sharing an edge.

3.4 Subdivided 3-graphs

For an *arbitrary* 3-graph $H = (V, E)$ and a sequence of natural numbers $\nu = (\nu_e : e \in E)$, construct a *subdivided* 3-graph $H'_\nu = (V', E')$ in the following way. The vertex set is $V' = V \cup V_E$, where $V_E = \bigcup_{e\in H} V_e$, $|V_e| = \nu_e$, and the sets V_e are pairwise disjoint and disjoint from V. The edge set E' is obtained by replacing each hyperedge $e = \{v_1, v_2, v_3\}$ with all triples of the form $\{v_i, v_j, v\}$, where $1 \le i < j \le 3$ and $v \in V_e$.

Fact 7. *For every H and ν, the hypergraph H'_ν does not contain any 3-comb, i.e. $H'_\nu \in \mathcal{H}_0$.*

Proof. Consider an arbitrary edge $f = \{v_i, v_j, v\}$ of H'_ν, where $v \in V_e$ and $e = \{v_1, v_2, v_3\}$. Every edge which intersects f at v has also another vertex common with f. Thus, f cannot intersect three disjoint edges of H'_ν, and so there is no 3-comb in H'_ν. □

Note that for a simple H, every matching $M = \{\{u_1, v_1\}, \ldots, \{u_t, v_t\}\}$ in the shadow graph $\Gamma(H)$ of H (obtained by replacing each hyperedge with a graph triangle) determines precisely $\prod_{i=1}^{t} \nu_{e_i}$ matchings of size t in H'_ν, where e_i is the unique edge of H which contains the pair $\{u_i, v_i\}$. Moreover, every matching of H'_ν is determined this way. Thus, for simple H, the problem of counting matchings in H'_ν reduces to counting matchings in graphs with weighted edges.

In the special case when for all $e \in H$ we have $\nu_e = 1$ (see Fig.3(b)), the above defined operation generalizes the operation of edge subdivision for graphs and, as for graphs, it preserves hypergraph planarity. For instance, consider a planar triangulation G and its triangle 3-graph $H = T(G)$. Clearly, H may contain lots of 3-combs. However, the subdivision H'_1 of H is free of 3-combs, but still represents a triangulation.

3.5 Enriched tight cycles

Recall the definition of a $(k-1)$-overlapping k-cycle $C_n^{(k)}(k-1)$ (a tight cycle) from Section 1.5 and observe that it is H_0-free and has precisely n edges. We will now "enrich" this tight cycle by increasing its number of edges by a factor of $\frac{k+1}{2}$, still keeping it in the family $\mathcal{H}_0$.

Fox $k \geq 3$, let $C = C_n^{(k+1)}(k-1)$ be a $(k-1)$-overlapping $(k+1)$-cycle on n vertices. We build a k-graph $D = D_n^{(k)}$ by replacing each edge e of C by the k-clique $\binom{e}{k}$. Note that $D_n^{(k)}$ contains $C_n^{(k)}(k-1)$ and has $\frac{k+1}{2}n$ edges, so, in a sense, it can be viewed as an enriched tight cycle.

Fact 8. *For all $k \geq 3$, the k-graph $D_n^{(k)}$ is H_0-free.*

Proof. Suppose that there is a copy of the 3-comb H_0 in $D_n^{(k)}$. Let us denote its pairwise disjoint edges by e_1, e_2, e_3, and the intersecting edge by e_4. Suppose that e_4 is contained in an edge e of C. There are only two other edges of C which intersect e and the total number of vertices in the edges of C intersecting e (including e itself) is at most $3k-1$. The edges e_1, e_2, e_3 must each belong to one of these edges of C. But this contradicts the fact that e_1, e_2, e_3 are mutually disjoint. □

3.6 Rooted blow-up hypergraphs

The families from subsections 3.2 and 3.3 consisted of k-graphs with only $o(|V|^{3/4})$ edges, while the H_0-free k-graphs constructed in subsections 3.1, 3.4 and 3.5 had $\Theta(|V|)$ edges. Here, we define H_0-free k-graphs with up to $\Theta(|V|^2)$ edges.

Partition an N-vertex set V into n nonempty sets $V_1, \ldots, V_n$, and fix one vertex $v_i \in V_i$ for each $i = 1, \ldots, n$. Next, for every pair $1 \leq i < j \leq n$ include to the edge set E the family E_{ij} of all k-element subsets of $V_i \cup V_j$ containing both, v_i and v_j.

Fact 9. *The k-graph $D = (V, E)$ is H_0-free.*

Proof. Suppose that there is a copy of the 3-comb H_0 in H. Let us denote its pairwise edges by e_1, e_2, e_3 and the intersecting edge by e_4. There exists a unique pair $1 \leq i < j \leq n$ such that $e_4 \in E_{ij}$. Then each of e_1, e_2, e_3 intersects the set $V_i \cup V_j$, and consequently contains either v_i or v_j. But this contradicts the fact that e_1, e_2, e_3 are mutually disjoint. □

Note that when $|V_i| = O(1)$ for all i, the hypergraph D has $\Theta(n^2)$ edges. In fact, when $|V_i| = 1$ and $k = 2$ we obtain the complete graph K_n.

4 Further Research

It remains an open question how to extend our result to larger classes of hypergraphs (perhaps even to the class of all uniform hypergraphs). The success in the

case of graphs relied mostly on the fact that every graph is free of 3-combs and thus $I \oplus F$ has a very simple structure. This is the case of the hypergraphs in the family $\mathcal{H}_0$ as well. For general hypergraphs, however, the presence of 3-combs may lead to situations where $e_1, e_2, e_3 \in I$, $e_4 \in F$, and $e_4 \cap e_i \neq \emptyset$, $i = 1, 2, 3$. Then, in the process of creating the canonical path $\gamma(I, F)$, even if we could somehow traverse the component Q of $I \oplus F$ containing the 3-comb e_1, e_2, e_3, e_4, in order to put e_4 on the current matching M_j we would need first to delete e_1 and e_2, and at least one of them, say e_2, by a transition of type (-). As e_2 might intersect two other (than e_4) edges of F, this may create a path of length three in the set $\eta_{M,M'}(I, F)$. This scenario can repeat many times and, consequently, $\eta_{M,M'}(I, F)$ may contain many (disjoint) paths of length three. Thus, the image of $\eta_{M,M'}$ would be much larger than $\Omega(H)$, not yielding (5).

Another direction of further research is to try to obtain an FPRAS for perfect matchings in *dense* k-uniform hypergraphs, where the density is measured as, e.g., in [14]. For $k = 2$ this was done in [11]. The 3-combs are an obstacle here too, but in addition, we are facing the problem of the necessity of including into the state space of the Markov chain matchings much smaller than the perfect ones (in [11] the state space consisted only of perfect and near-perfect matchings, that is, matchings missing just two vertices).

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