# Exact and Approximation Algorithms for Geometric and Capacitated Set Cover Problems* (Revised Version)

Piotr Berman[†], Marek Karpinski[‡], and Andrzej Lingas[§]

**Abstract.** First, we study geometric variants of the standard set cover motivated by assignment of directional antenna and shipping with deadlines, providing the first known polynomial-time exact solutions.
Next, we consider the following general (non-necessarily geometric) capacitated set cover problem. There is given a set of elements with real weights and a family of sets of the elements. One can use a set if it is a subset of one of the sets in the family and the sum of the weights of its elements is at most one. The goal is to cover all the elements with the allowed sets.
We show that any polynomial-time algorithm that approximates the uncapacitated version of the set cover problem with ratio $r$ can be converted to an approximation algorithm for the capacitated version with ratio $r + 1.357$.
The composition of these two results yields a polynomial-time approximation algorithm for the problem of covering a set of customers represented by a weighted $n$-point set with a minimum number of antennas of variable angular range and fixed capacity with ratio 2.357.

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† Department of Computer Science and Engineering, Pennsylvania State University. Email: berman@cse.psu.edu

‡ Department of Computer Science, University of Bonn. Email: marek@cs.uni-bonn.de

§ Department of Computer Science, Lund University. Research supported in part by VR grant 621-2008-4649. Email: Andrzej.Lingas@cs.lth.se

## 1 Introduction

In this paper, we study special geometric set cover problems and capacitated set cover problems.

In particular, the shapes of geometric sets we consider correspond to those of potential directional antenna ranges. Several geometric covering problems where a planar point set is to be covered with a minimum number of objects of a given shape have been studied in the literature, e.g., in [5, 6, 9, 12].

On the other hand, a capacitated set cover problem can be seen as a generalization of the classical bin packing problem (e.g., see [7]) to include several types of bins. Thus, we are given a set of elements $\{1, \ldots, n\}$, each with a demand $d_i$, a set of subsets of $\{1, \ldots, n\}$ (equivalently, types of bins), and an upper bound $d$ on set capacity. The objective is to partition the elements into a minimum number of copies of the subsets (bins) so the total demand of elements assigned to each set copy does not exceed $d$.

Capacitated set cover problems are useful abstraction in studying the problems of minimizing the number of directional antennas. The use of directional antennas in cellular and wireless communication networks steadily grows [2, 17, 19, 18]. Although such antennas can only transmit along a narrow beam in a particular direction they have a number of advantages over the standard ones. Thus, they allow for an additional independent communication between the nodes in parallel [18], they also attain higher throughput, lower interference, and better energy-efficiency [2, 17, 19].

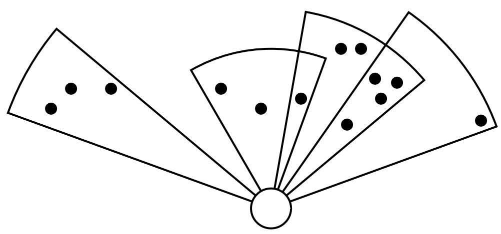

**Fig. 1.** The sectors correspond to the reaches of directional antennas.

We consider the following problem of optimal placement of directional antennas in wireless networks.

There is a base station coupled with a network infrastructure. The station transfers information to and from a number of customers within the range of directional antennas placed at this station. Each customer has fixed position and demand on the transmission capacity. The demands are unsplittable, thus a customer can be assigned only to a single antenna. One can choose the orientation and the angular range of an antenna. When the angular range is narrower an antenna can reach further so the area covered by any antenna is always the same. There is a common limit on the total bandwidth demand that can be assigned to an antenna. The objective is to minimize the number of antennas.

Berman et al. [3] termed this problem as MinAntVar and provided an approximation polynomial-time algorithm with ratio 3. They also observed in [3] that even when the angular range of antennas is fixed, MinAntVar cannot be approximated in polynomial time with ratio smaller than 1.5 by a straightforward reduction from Partition (see [11]).

We provide a substantially better polynomial-time approximation algorithm for MinAntVar achieving the ratio of 2.357. Our algorithm is based on two new results which are of independent interest in their own rights.

The first of these results states that a cover of the set of customers with the minimum number of antennas without the demand constraint can be found in polynomial time. Previously, only a polynomial-time approximation with ratio 2 as well as an integrality gap with set cover ILP were established for this problem in [3].

The second result shows that generally, given an approximate solution with ratio $r$ to an instance of (uncapacitated) set cover, one can find a solution to a corresponding instance of the capacitated set cover, where each set has the same capacity, within $r + 1.357$ of the optimum. This result is especially useful when applied to variants of set cover whose uncapacitated versions admit close approximations or even exact algorithms running in polynomial time, e.g., variants of geometric set cover [5, 6, 9, 12] or in particular MinAntVar.

Berman et al. considered also the following related problem which they termed as BinSchedule [3]. There is a number of items to be delivered. The $i$-th item has a weight $d_i$, arrival time $t_i$ and patience $p_i$, which means that it has to be shipped at latest by $t_i + p_i$. Given a capacity of a single shipment, the objective is minimize the number of shipments.

Similarly as Berman et al. could adopt their approximation for MinAntVar to obtain an approximation with ratio 3 for BinSchedule [3], we can adopt our approximation for MinAntVar to obtain a polynomial-time approximation algorithm with ratio 2.357 for BinSchedule.

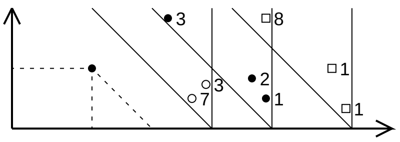


**Fig. 2.** The $X$ coordinate of an item $i$ encodes $t_i$ and the $Y$ coordinate encodes $p_i$. Shipment has capacity 10. The numbers indicate the weights. Items which are to be shipped together must be enclosed by an angle.

Our third main result is a PTAS for a dual problem to capacitated set cover where the number of sets (e.g., antennas) to use is fixed and the task is to minimize the maximum set load, in case the sets correspond to line intervals or arcs. In the application to directional antennas, the aforementioned correspondence comes from fixing the radius and hence also the angular range of the antennas and the problem has been termed as MinAntLoad in [3]. The task is to minimize the maximum load of an antenna. In [3], there has been solely presented a polynomial-time approximation with ratio 1.5 for MinAntLoad.

Marginally, we also discuss the approximability of the generalization of MinAntVar to include several bas stations for antennas, and in particular show its APX-hardness already in the uncapacitated case.

**Organization:** In Section 2 we present problem definitions and notations. In Section 3, we derive our polynomial-time dynamic programming method for the uncapacitated variant of MinAntVar. In Section 4, we show our general method of the approximate reduction of the capacitated vertex cover to the corresponding uncapacitated one. By combing it with the method of Section 3, we obtain the 2.357 approximation for MinAntVar. Next, in Section 5, we present the PTAS for MinAntLoad, or more generally, for minimizing the maximum load in capacitated set cover of bounded cardinality, in case the sets correspond to intervals or arcs. In the final section, we briefly discuss the approximability of the multi base-station generalization of MinAntVar.

## 2 Preliminaries

This section presents terminology and notation used throughout this paper.

We use $U$ to denote $\{1, 2, \ldots, n\}$. If $x_i \in \mathbb{R}$ are defined for $i \in U$ (e.g., $d_i$) and $A \subset U$ then $x(A) = \sum_{i \in A} x_i$ (e.g.,$d(A) \sum_{i \in A} d_i$).

An instance of the set cover problem is given by a family $\mathcal{S}$ of subsets of $U = \{1, \ldots, n\}$. A cover is $\mathcal{C} \subset \mathcal{S}$ such that $\bigcup_{A \in \mathcal{C}} A = U$. The objective is to minimize $|\mathcal{C}|$. An instance of capacitated set cover also specifies $d_i$ for $i \in U$. A capacitated cover is a family of sets $\mathcal{C}$ satisfying (i) for each $A \in \mathcal{C}$ there exists $B \in \mathcal{S}$ s.t. $A \subset B$, while $d(A) \leq 1$, and (ii) $\bigcup_{A \in \mathcal{C}} A = U$. Again, the objective is to minimize $|\mathcal{C}|$.

For each $j \in U$, we denote its radial coordinates by $(r_j, \theta_j)$, where $r_j$ stands for the radius and $\theta_j$ for the angle. We define an angle sector with radius bound $\delta$ as

$$\mathcal{R}(r, \alpha, \delta) = \{j \in U : \; r_j \leq r \text{ and } \theta_j = \alpha + \beta \text{ with } 0 \leq \beta \leq \delta\}.$$

In MinAntVar as well as in its uncapacitated variant, $U$ is the set of customers with radial coordinates defined in respect to the position of the base station. This is a variant of capacitated (or uncapacitated) set cover where $\mathcal{S}$ consists of sets of customers that can be within range of a single antenna, *i.e.* of the form $\mathcal{R}(r, \alpha, \rho(r))$, where $\rho(r)$ is the angular width of an antenna with radial reach $r$.

The trade-off function $\rho$ is decreasing; to simplify the proofs, we assume that $\rho(r) = 1/r$. We can change the $r$-coordinates to obtain exactly the same family of antenna sets as for arbitrary $\rho$.

## 3 Uncapacitated Cover by Antenna Sets

To simplify proofs, we will ignore the fact that the radial coordinate has a "wrap-around". We also renumber the customers so $\theta_i < \theta_{i+1}$ for $1 \leq i < n$. Observe that if $\theta_i = \theta_j$ and $r_i \geq r_j$ then every antenna set that contains $i$ also contains $j$, so we can remove $j$ from the input.

It suffices to consider only $n(n+1)/2$ different antenna sets. For such an antenna set $A$, let $i = \min A$, $j = \max A$. If $i = j$, we denote $A$ as $A[i,i] = \{i\}$, and if $i < j$, we set $r(i,j) = (\theta_j - \theta_i)^{-1}$ and define $A[i,j] = \mathcal{R}(r(i,j), \theta_i, 1/r(i,j))$. (This definition is more complicated when the "wrap-around" is allowed.) Because $A \subseteq A[i,j]$ we can use $A[i,j]$ in our set cover instead of $A$.

We say that points $i$ and $j$ are compatible, denoted $i \heartsuit j$, if $i \leq j$ and there exists an antenna set that contains $\{i,j\}$. If $i = j$ then $i \heartsuit j$ is obvious; if $i < j$ then $i \heartsuit j$ is equivalent to $\{i,j\} \subseteq A[i,j]$ which in turn is equivalent to $r_i, r_j \leq r(i,j)$. If $i \heartsuit j$, we set $S[i,j] = \{k : \; i \leq k \leq j\} \setminus A[i,j]$.

We solve our minimum cover problem by dynamic programming. Our recursive subproblem is specified by a compatible pair $i, j$ and its objective is to compute the size $C[i,j]$ of minimum cover of $S[i,j]$ with antenna sets. If we modify the input by adding the points 0 and $n+1$ with coordinates $(\varepsilon, \theta_1 - 1)$ and $(\varepsilon, \theta_n + 1)$ then our original problem reduces to computing $C[0, n+1]$.

If $S[i,j] = \varnothing$ then $C[i,j] = 0$. Otherwise, $S[i,j] = \{a_0, \ldots, a_{m-1}\}$, where $a_k < a_{k+1}$ for $k = 0, \ldots, m-2$.

We define a weighted graph $G_{i,j} = (V_{i,j}, E_{i,j}, c)$, where $V_{i,j} = \{0, \ldots, m\}$, $(k, \ell+1) \in E_{i,j}$ iff $a_k \heartsuit a_\ell$ and for an edge $(k, \ell+1)$, we define the cost $c(k, \ell+1) = 1 + C[a_k, a_\ell]$.

Note that $G_{i,j}$ is acyclic. Therefore, we can find a shortest (i.e., of minimum total cost) path from 0 to $m$ in time $O(|E_{i,j}|) = O(n^2)$ [8]. Let $d$ be the length of this path. We will argue that $C[i,j] = d$.

First, we show a cover of $S[i,j]$ with $d$ antenna sets. A path from 0 to $m$ in $G_{i,j}$ is an increasing sequence, and a path edge $(u,v)$ with cost $c$ corresponds to a cover of $\{a_u, a_{u+1}, \ldots, a_{v-1}\}$ with $A[a_u, a_{v-1}]$ and $c-1$ antenna sets that cover $S[a_u, a_{v-1}]$.

Conversely, given a cover $\mathcal{C}$ of $S[i,j]$, we can obtain a path with cost $|\mathcal{C}|$ in $G_{i,j}$ that connects 0 with $m$.

For $A[k,\ell] \in \mathcal{C}$, we say that $\ell - k$ is its *width*. To make a conversion from a cover $\mathcal{C}$ of $S[i,j]$ to a path in $G_{i,j}$, we request that $\mathcal{C}$ has the minimum sum of widths among the minimum covers of $S[i,j]$.

This property of $\mathcal{C}$ implies that if $A[k,\ell] \in \mathcal{C}$ then:

$k, \ell \in S[i,j]$,

$k$ and $\ell$ are not covered by $\mathcal{C} - \{A[k,\ell]\}$ (otherwise we eliminate $A[k,\ell]$ from $\mathcal{C}$ or replace it with a set that has a smaller width).

Note that for each pair of sets $A[k,\ell], A[k',\ell'] \in \mathcal{C}$, where $k < k'$, one of two following cases applies:

1. $\ell < k'$, i.e., $A[k,\ell]$ precedes $A[k',\ell']$;
2. $\ell' < \ell$, i.e., $A[k',\ell']$ is nested in $A[k,\ell]$.

Let $\mathcal{D}$ be the family of those sets in $\mathcal{C}$ that are not nested in others. Clearly $\mathcal{D}$ can be ordered by the leftmost elements in the sets. Note that if $A[k,\ell] \in \mathcal{D}$ then for some $f, g, c$, we have

$a_f = k \in S[i,j]$,
$a_g = \ell \in S[i,j]$,
$c-1$ sets of $\mathcal{C}$ are nested in $A[k,\ell]$ and they cover $S[i,j]$,
$(f, g+1)$ is an edge in $G_{i,j}$ with cost $c$,
$g+1 = m$ or $A[a_{g+1}, \ell'] \in \mathcal{D}$ for some $\ell'$.

These $(f, g+1)$ edges form a path that connects 0 with $m$ with cost $|\mathcal{C}|$.

Our dynamic programming algorithm solves the $n(n+1)/2$ subproblems specified by compatible pairs $i, j$ in a non-decreasing order of the differences $j - i$. In the reduction of a subproblem to already solved subproblems the most expensive is the construction of the graph $G_{i,j}$ and finding the shortest path in it, both take quadratic time. Hence, we obtain our main result in this section.

**Theorem 1** *The uncapacitated version of the problem of minimum covering with antenna sets n points, i.e., the restriction of* MinAntVar *to the case where all point demands are zero, is solvable in time $O(n^4)$ and space $O(n^2)$* [1]*.*

Previously, only a polynomial-time approximation algorithm with ratio two was known for the uncapacitated version of MinAntVar [3].

## 4 From Set Cover to Capacitated Set Cover

By the discussion in the previous section, it is sufficient to consider only $O(n^2)$ antenna sets in an instance of MinAntVar on $n$ points. Hence, MinAntVar is a special case of minimum capacitated set cover.

Since we can determine a minimum uncapacitated set cover of an instance of MinAntVar by ignoring the demands and running the dynamic programming method given in the previous section, we shall consider the following more general situation.

We are given an instance of the general problem of minimum capacitated set cover and an approximation with ratio $r$ for minimum set cover of the corresponding instance of minimum set cover obtained by removing the demands. The objective is to find a good approximation of a minimum capacitated set cover of the input instance.

We can obtain an approximation with ratio $r + 1.692$ for minimum capacitated set cover on the base of an approximation with ratio $r$ for minimum uncapacitated set cover $\mathcal{U}^*$ by running a simple greedy FFD algorithm (see Fig. 3). Our analysis of this algorithm in part resembles that of the first-fit heuristic for bin-packing [7, 10], but the underlying problems are different. It yields the approximation ratio $r + 1.692$. By refining the algorithm and its analysis, we can improve the factor substantially to $r + 1.357$. Because of space considerations for the proof the reader is referred to the full version.

[1] Very recently, M. Patrascu found a tricky way of improving the time complexity of an equivalent problem to a cubic one [16].

```
𝒬 ← ∅
for (U ∈ 𝒰*)
    while (U ≠ ∅)
        Q ← ∅
        for (i ∈ U, with d_i non-decreasing)
            if (d(Q) + d_i ≤ 1)
                insert i to Q
                remove i from U and P
        insert Q to 𝒬
```

**Fig. 3.** FFD, First Fit Decreasing algorithm for converting a cover into a capacitated cover.

**Theorem 41** *Let an instance of capacitated set cover be specified by a universe set $P = \{1, ..., n\}$, demands $d_i \geq 0$ for each $i \in P$, and a family $\mathcal{S}$ of subsets of $P$. If an approximation with ratio $r$ for minimum set cover of the uncapacitated version of the instance (i.e., where the demands are removed) is given then a capacitated set cover of the input instance of size at most $r + 1.357$ times larger than the optimum can be determined in polynomial time.*

**Corollary 42** *There exists a polynomial-time approximation algorithm for the problem of* MinAntVar *with ratio* 2.357.

By the reduction of BinSchedule to MinAntVar given in [3], we also obtain the following corollary.

**Corollary 43** *There exists a polynomial-time approximation algorithm for the problem of* BinSchedule *with ratio* 2.357.

## 5 PTAS for MinAntLoad

In MinAntLoad problem, the radius of antennas is fixed and the number $m$ of antennas that may be used is specified. The task is to minimize the maximum load of an antenna. A polynomial-time approximation for this problem achieving ratio 1.5 is presented in [3].

In the dual problem MinAnt, the maximum load is fixed and the task is to minimize the number of antennas. Recall that achieving an approximation ratio better than 1.5 for the latter problem requires solving the following problem equivalent to Partition.

Suppose that all demands can be covered with a single set, the load threshold is $D$ and the sum of all demands is to $2D$. Decide whether or not two antennas are sufficient (which holds if and only if one can split the demands into two equal parts).

However, in case of the corresponding instance of MinAntLoad, we can apply FPTAS for the SubsetSum problem [14] in order to obtain a good approximation for the minimization of the larger of the two loads.

If all demands can be covered by a single antenna set (and the sum of demands is arbitrary) then MinAntLoad problem is equivalent to that of minimizing the makespan while scheduling jobs on $m$ identical machines. Hochbaum and Shmoys showed a PTAS for this case in [13].

Interestingly enough, the PTAS of Hochbaum and Shmoys can be modified for MinAntLoad, while it does not seem to be the case with their practical algorithms that have approximation ratios of 6/5 and 7/6 [13].

**Theorem 2** MinAntLoad *for $n$ points admits an approximation with ratio $1+\varepsilon$ in time $n^{\frac{1}{\varepsilon}\ln\frac{1}{\varepsilon}+O(1)}$.*

**Proof**. See the full version. ❑

Note that the only geometric property of antennas with fixed radius that we used to design the PTAS for MinAntLoad is their correspondence to intervals or arcs. Hence, we obtain the following generalization of Theorem 2.

**Theorem 3** *The problem of minimizing the maximum load in a capacitated set cover where the sets correspond to intervals or arcs admits a PTAS.*

## 6 Extentions to Multi-Base MinAntVar

Our general method of approximating with ratio $r + 1.357$ minimum capacitated set cover on the base of an approximate solution with ratio $r$ to the corresponding minimum (uncapacitated) set cover can be also used to approximate optimal solutions to the natural extension of MinAntVar to include several base stations. It is sufficient to combine it with known approximation algorithms for geometric set cover, e.g., [5, 6, 12]. In this way, we can obtain an approximation with ratio $O(\log OPT)$ for the multi-base variant of MinAntVar, where $OPT$ is the size of minimum uncapacitated set cover with antennas, see the full version. We can also prove the APX-hardness of the multi-base uncapacitated variant of MinAntVar by a reduction from a Minimum Line Covering Problem [4], see the full version.

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