Cycles and Paths in Edge-Colored Graphs with Given Degrees

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Abstract

Sufficient degree conditions for the existence of properly edge-colored cycles and paths in edge-colored graphs, multigraphs and random graphs are inverstigated. In particular, we prove that an edge-colored multigraph of order n on at least three colors and with minimum colored degree greater than or equal to $\lceil \frac{n+1}{2} \rceil$ has properly edge-colored cycles of all possible lengths, including hamiltonian cycles. Longest properly edge-colored paths and hamiltonian paths between given vertices are considered as well.

1 Introduction and Notation

In this work, we consider sufficient degree conditions guarantying the existence of colored cycles and paths in graphs whose edges are colored with any number of colors. The study of spanning subgraphs with specified color patterns in edge-colored graphs has witnessed significant developments over the last decade, and this from both theoretical and practical perspectives. In particular, problems arising in molecular biology are often modeled by means of colored graphs, i.e., graphs with colored edges and/or vertices [17]. Given such a graph, original problems correspond to extracting subgraphs such as Hamiltonian and Eulerian paths or cycles colored in a specified pattern [16, 17]. The most natural pattern in such a context is that of a proper coloring, which entails adjacent edges/vertices having different colors. Properly colored paths and cycles have applications in various other fields, as in VLSI for compacting a programmable logical array [15]. Although a large body of work has already been done [3, 4, 5, 6, 8, 18], in most of that previous work the number of colors was restricted to two. For instance, while it is well known that properly edge-colored hamiltonian cycles can be found efficiently in 2-edge colored

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complete graphs, it is a long standing question whether there exists a polynomial algorithm for finding such hamiltonian cycles in edge-colored complete graphs with three colors or more [6]. Notice that the hamiltonian path problem was solved recently in [11] in the case of complete graphs, whose edges are colored with an arbitrary number of colors. Recent work on cycles and paths involving colored degrees in edge-colored graphs are found in [12, 13].

Formally, let $\{1, 2, \cdots, c\}$ be a set of given $c \geq 2$ colors. Throughout the paper, G^c denotes an edge-colored multigraph so that each edge is colored with some color $i \in \{1, 2, \cdots, c\}$ and no two parallel edges joining the same pair of vertices have the same color. The vertex and edge-sets of G^c are denoted by $V(G^c)$ and $E(G^c)$, respectively. The order of G^c is the number n of its vertices. For a given color i, $E^i(G^c)$ denotes the set of edges of G^c on color i. When no confusion arises, we write V, E and E^i instead of $V(G^c), E(G^c)$ and $E^i(G^c)$, respectively. When G^c is not a multigraph, i.e., no parallel edges between any two vertices are allowed, we call it a graph, as usual. For edge-colored complete multigraphs, we write K_n^c instead of G^c. If H is a subgraph of G^c, then $N^i(x, H)$ denotes the set of vertices of H, joined to x with an edge in color i. Whenever $H \cong G^c$, for simplicity, we write $N^i(x)$ instead of $N^i(x, G^c)$. The colored $i-degree$ of x, denoted by $d^i(x)$ equals $|N^i(x)|$, i.e., the cardinality of $N^i(x)$. For a given vertex x and a given positive integer k, the inequality $d^c(x) \geq k$ means that for every $i \in \{1, 2, \cdots, c\}$, $d^i(x) \geq k$. The edge between the vertices x and y is denoted by xy, and its color by $c(xy)$. If A_1 and A_2 are vertex-disjoint subsets of V, then the set of edges between A_1 and A_2 is denoted by A_1A_2, while the set of edges among the vertices of A_1 is denoted by A_1A_1. A subgraph of G^c is said to be properly edge-colored, if any two adjacent edges in this subgraph differ in color. The length of a path is the number of its edges. A matching M of G^c is a subset of $E(G^c)$ such that no two edges in M share a common vertex. It is perfect when its cardinality is $\frac{n}{2}$. For a given color i, M_i denotes a monochromatic matching on color i. An edge-colored multigraph G^c of order n is called pancyclic if it contains properly edge-colored cycles of all possible lengths $2,\ 3,\ 4,\ 5, \cdots, n$. Similarly, G^c is even-pancyclic if it contains properly edge-colored cycles of all possible even lengths $2,\ 4,\ 6,\ 8, \cdots, 2\left\lfloor \frac{n}{2} \right\rfloor$.

The paper is organized as follows: In Section 2 we study properly edge colored cycles and paths for edge-colored graphs. In Section 3 paths and cycles in edge-colored multigraphs are concerned. Some concluding remarks are given in Section 4.

2 Graphs

Let us start with a theorem concerning properly edge-colored paths in edge-colored graphs of minimum colored degree d.

Theorem 2.1. *Let G^c be a 2-edge colored graph such that for every vertex x, $d^i(x) \geq d \geq 1$, $i \in \{1, 2\}$. Then G^c has a properly edge-colored path of length at least $2d$.*

Proof. To ease discussion of the notions to come, we suppose that the two colors used are red and blue. Now, we will introduce some further notation with a scope limited to this proof only. For any properly edge-colored cycle C and any edge uv with $u \notin V(C)$ and $v \in V(C)$, there is only one way (either clockwise or counterclockwise in a plane drawing of C) in which we can proceed along C using the edge uv while keeping the alternating pattern. We denote the resulting properly edge-colored path of length $|C|$ by uvC.

Assume now, for a contradiction, that the conclusion of the theorem does not hold. Let then P be a longest properly edge-colored path of length $r < 2d$. We need not specify that $2d$ is less than or equal to n, since the graph is simple. Let x and y be the two endpoints of P. We call an edge xz of G^c (resp. yz) *external* if it is colored otherwise than is the unique edge of P incident with x (resp. y). Clearly, no edge xz or yz with $z \in V(G) \setminus V(P)$ is external. Moreover, the number of external edges is at least $2d$. Extensive use will be made throughout the proof of the following observation: if a vertex u is the endpoint of two properly edge-colored paths P_1, P_2 of maximum length in G^c, and if the two paths start with edges colored differently at u, then all the neighbours of u (whether red or blue) are in $P_1 \cup P_2$. Therefore $P_1 \cup P_2$ has length at least $2d$. If, in addition, both paths happen to be included in P, then we are home, because then P will be $2d$ in length at least. First, we can see that $V(P)$ does not contain any properly edge-colored cycle of length $r+1$ since, from the degree condition, we can use any vertex u of P to exit the cycle, which yields an even longer path than P. Furthermore, the previous observation implies that $V(P)$ does not contain cycles of length r either, since that would imply the existence of a vertex $t \in V(P)$ that is linked with two edges in different colors to a cycle of length r, an occurence which would cause vertex t to be the enpoint of two paths P_1 and P_2 of length r, both included in $V(P)$ and starting from t with edges colored differently: we know that this entails $V(P)$ being $2d$ in length.

Now, with those preliminary remarks in mind, we claim that:

Assertion 1: There exists a partition of $V(P)$ either into two properly edge-colored cycles or into two properly edge-colored cycles C_1 and C_2 and a vertex u. Moreover, in the latter case, u is linked to the two cycles with two edges of different colors.

Proof of the Assertion 1:

We distinguish two cases depending on the parity of r.

Case 1: r is odd.

Set then $P : x_1y_1x_2 \cdots x_py_p$ for some integer $p \geq 1$, so that $x = x_1$ and $y = y_p$ and $r = 2p-1$. Suppose that x_1y_1 is red. Observe that every external edge is incident with one endpoint of some blue edge y_ix_{i+1}, $i \leq p-1$. Since there are $2d > 2(p-1)$ external edges at least and only $p-1$ blue edges y_ix_{i+1}, $i \leq p-1$, in P, there is at least one edge y_ix_{i+1} that is incident with three or more external edges. Then, either the pair of edges $\{x_1x_{i+1}, y_iy_p\}$ are both external, or the pair of edges $\{x_1y_i, x_{i+1}y_p\}$ are. Therefore, either the cycle of length r: $x_1y_1...x_iy_px_py_{p-1}...y_{i+1}x_1$ is properly edge-colored , which is impossible according to the above, or the cycles $x_1y_1x_2y_2...x_ix_1$ and $y_{i+1}x_{i+1}...x_py_py_{i+1}$ form a partition of $V(P)$ into properly edge-colored cycles, as claimed.

Case 2: r is even.

Set then $P : x_1y_1x_2 \cdots x_py_px_{p+1}$, with $p \leq d-1$, and suppose that x_1y_1 is red. For every vertex y_i, $i \leq p-1$, one of the edges x_1x_{i+1}, x_ix_{p+1} must not be external, otherwise the cycle

$x_1x_{i+1}y_{i+1}x_{i+2}...x_{p+1}x_i...x_1$ would be a properly edge-colored cycle of length r, which is not possible according to the observation above. Similarly, for every vertex x_j, $j > 1$, either one of the edges x_1y_{j-1}, y_jx_{p+1} is not external, or vertex x_j together with the cycles $x_1y_{j-1}...y_1x_1$ and $y_jx_{j+1}...x_{p+1}y_j$ form a partition as in the assertion. Observe furthermore that none of the pairs of vertices x_1y_1 and y_px_{p+1} forms an external edge since the graph is simple. Therefore, if our assertion were not true, the number of pairs of vertices within P that do not form external edges would be at least $2(p-1)+2 = 2p$ in number. That would leave us with no more than $2p-1$ external edges from among the $2(2p)-1$ potential ones, which are incident with either x or y (excluding xy). That amounts to fewer than $2d$ external edges, a contradiction. The assertion is proved.

Returning to the proof of the theorem, we consider now a partition as given in Assertion 1. The smaller of the two cycles in the partition is denoted by C_1. An edge uv is called *alien* if one of its endpoints is in C_1 whereas the other one is in $G^c - P$. Now we distinguish two cases depending on the parity of r again. The facts proved under each case will be denoted in reference to their respective cases lest confusion may arise.

Case A: r is odd.

Let $C_1 = x_1x_2\cdots x_kx_1$ and $C_2 = y_1y_2\cdots y_t$ be two properly edge-colored cycles that partition $V(P)$ as in Assertion 1. Thus $k \leq d$. Since the two cyles are mergeable into P, there is at least one edge between C_1 and C_2. We may suppose that x_1y_1 is that edge.

We may suppose as well that x_1y_1 and x_1x_2 are blue. Thus, the path $x_2x_3\cdots x_kx_1y_1C_2$ is properly edge-colored of length r. Therefore, there is no blue alien edge incident with x_2 (or a longer path would result from that). We have thus proved the first fact:

Fact A1: There is no blue alien edge incident with x_2.

Therefore, as the blue degree of x_2 is greater than $k-1$, there is at least one blue edge x_2y_j, for some $j \in \{1.\cdots t\}$. Now observe this fact:

Fact A2: There is no red alien edge incident with x_3.

The existence of such a red alien edge would imply that some vertex u in $G^c - P$ was part both of a red edge x_3u and of another blue edge uv, with $v \notin C_1$. The existence of v is guaranteed by the red degree of u being greater than $k-1$. Now, if $v \notin C_2$, we get the path of length $r+1$: $vux_3x_4\cdots x_1y_1C_2$, a contradiction. On the other hand, if $v = y_q \in C_2$, we get the path of length $r+1$: $x_2x_1x_k\cdots x_3uy_qC_2$, a contradiction again.

From Fact A2 together with our assumption that $k \leq d$, we conclude that there is a red edge in the form x_3y_i. Then the path $P' = x_2x_1x_kx_{k-1}\cdots x_3y_iC_2$ is another path of length r with x_2 as endpoint, with a different color incident with x_2. This, as we have seen, is proof that P has length $2d$ at least, which settles the case.

Case B: r is even.

Let $C_1 = x_1x_2\cdots x_kx_1$, $C_2 = y_1y_2\cdots y_t$ and u_1 be the two properly edge-colored cycles and the singleton, respectively, that partition $V(P)$ as in the proof of Assertion 1. Assume that C_1 is the smaller cycle of the two , i.e., $k \leq d$. Furtehrmore, assume that u_1 is linked to C_1 with a red edge, say,u_1x_1, and to C_2 with a blue edge, say u_1y_1. Suppose without loss of generality that x_1x_2 is blue and y_1y_2 is red. Hence the properly edge-colored path of length r: $x_kx_{k-1}\cdots x_1uy_1C_2$. Observe that this path starts from x_k with a blue edge. Therefore, there is no alien red edge incident with x_k, otherwise a longer properly edge-colored path would result from it. We have just proved:

Fact B1: There is no alien red edge incident with x_k.

Now, since the red degree of x_k is greater than $k-1$ and x_kx_{k-1} is blue, there is at least one red edge in the form x_ky_i. Now, we claim that:

Fact B2: There is no alien blue edge incident with x_{k-1}.

The proof being similar to that of Fact A2, we will give only a sketch of it. Suppose that we have a blue edge $x_{k-1}v$, with $v \notin V(P)$. From the red degree condition on v together with the fact that neither vx_k nor vx_{k-1} are red, there must be some red edge vw, with $w \notin V(C_1)\cup\{u_1\}$. Now, either $w \notin V(C_2)$ or $w \in V(C_2)$. In the first case, we get a properly edge-colored path of length $r+1$ (namely: $wvx_{k-1}\cdots x_1u_1y_1C_2$), while in the second case we get another path P' of length r with x_k as enpoint and starting from x_k with a red edge (as opposed to a blue edge for P): $x_kx_1\cdots x_{k-1}wC_2$. Thus both cases lead to a contradiction, according to our observation above, and the fact is proved.

We conclude from Fact B2 that there is at least one blue edge $x_{k-1}y_j$. Now, if no alien blue edge is incident with x_k, we are done, because that would mean, in view of Fact B1, that all the neighbors of x_k (of which there are $2d$ at least) are in P, a contradiction. Thus, we may suppose that some edge x_ku_k ($u_k \in V(G^c)\diagdown V(P)$) is blue. Observe, furthermore, that u_k cannot have a red neighbour outside P because such a neighbour z, if it existed, would yield the following properly edge-colored path of length $r+1$: $zu_kx_kx_1\cdots x_{k-1}wC_2$. From the degree condition on u_k, we conclude that there must be some red neighbour z_j of u_k in C_2.

Let us recap what we have obtained thus far. We started with vertex u_1 and, proceeding backward along the smaller cycle, we concluded with the existence of a similar vertex u_k. Now, if we Repeat the same steps k times over, we get Fact B3, which sums up our findings so far:

Fact B3: For every vertex x_i of C_1, there are two vertices $u_i \in V(G^c)\diagdown V(P)$ and $z_i \in C_2$ such that:

(1) if i is odd, then x_iu_i is a red edge of G^c and u_iz_i is a blue edge.

(2) if i is even, then x_iu_i is a blue edge of G^c and u_iz_i is a red edge.

It should be emphasized here that none of B3(1) and B3(2) contradict A1 and A2 in any way, since those are obviously non-overlapping cases, as clearly suggested by their notation.

Now, set $X = \{x_i | i = 1 \ mod\ 2\}$, and $Y = \{x_i | i = 0 \ mod\ 2\}$. Observe that every vertex x_i of X is the endpoint of a longest path starting from x_i with a red edge. Similarly, every vertex x_i of Y is the endpoint of a longest path starting from x_i with a blue edge. Observe that no blue edge x_ix_j has both its endpoints in X, because that would yield the properly edge-colored path: $x_{i+1}x_{i+2}\cdots x_{j-1}x_jx_ix_{i-1}\cdots x_{j+1}u_{j+1}z_{j+1}C_2$, which has length r and starts from $x_{i+1} \in Y$ with a red edge. As x_{i+1} starts another path of length r with a blue edge incident with x_{i+1}, we conclude that x_{i+1} has $2d$ neighbors in P,a contradiction. Similarly,Y does not have any red edge. Moreover, there is no blue edge between X and $G^c - P$, and no red edge between Y and

$G^c - P$ (since any of those edges would extend a longest path).

Thus, every vertex x_i of X has not more than $|Y| - 1 = \frac{k}{2} - 1$ blue edges within C_1, which accounts for the fact that all edges x_iX are red and one edge at least from x_iY is red. Moreover, there are no blue edges at all between X and $G^c - P$. Similarly, every vertex x_j of Y has no more than $|X| - 1 = \frac{k}{2} - 1$ red edges within C_1, and there are no red edges at all between Y and $G^c - P$.

On the other hand, every vertex x_j in Y has at least d edges within P, with $d \geq \frac{r}{2} = \frac{k+t}{2}$. Hence, every vertex x_j of Y has a least $\frac{|C_2|}{2} = \frac{t}{2}$ red edges with C_2, whereas x_1 has at least $\frac{t}{2} + 1$ blue edges with C_2 (because x_1u_1 is red). Hence the fact again:

Fact B4: There is a vertex y_i of C_2 such that one of the two conditions holds:

(i) x_1y_{i+2} is a blue edge, x_2y_i is a red edge and i is odd

(ii)x_1y_i is a blue edge, x_2y_{i+2} is a red edge and i is even.

Before proceeding with the proof, notice that each of those conditions, if established, would yield a properly edge-colored path of length $r+1$, proving the theorem. In case (i), for instance, that path would be: $u_1x_1y_{i+2}y_{i+3}\cdots y_ix_2x_3\cdots x_ku_k$. If $u_k = u_1$, that path is a properly edge-colored cycle of length r. Now, let us prove the fact.

We call any portion of length 2 on the cycle C_2 a 2-segment. For any 2-segment $s = y_iy_{i+1}y_{i+2}$, let us say that a pair uv is consistent with s if it is any one of the edges arising in the conditions of the fact. More formally, a pair uv is consistent with $s = y_iy_{i+1}y_{i+2}$ if one the four conditions holds:

(1) $u = x_1$, $v = y_{i+2}$, uv blue and i odd

(2) $u = x_1$, $v = y_i$, uv blue and i even

(3) $u = x_2$, $v = y_{i+2}$, uv red and i even

(4) $u = x_2$, $v = y_i$, uv red and i odd

The conditions are redily seen to be exclusive.

Notice that a blue edge x_1y_j is consistent with only one 2-segment. Similarly, a blue edge x_2y_j is consistent with one 2-segment exactly. Consider the function such that $1(s,e) = 1$ if the edge e is consistent with s, and $1(s,e) = 0$ otherwise. For any s, denote by $|s|$ the number of pairs consistent with s. Now, summing the terms $1(s,e)$ in two different ways, we get: $\sum_s \sum_e 1(s,e) = \sum_s |s| = d^b_{C_2}(x_1) + d^r_{C_2}(x_2) > |C_2| = t$. Hence, one s at least has two consistent pairs, which proves the fact and the theorem. □

Corollary 2.2. *Let G^c be a c-edge colored graph, $c \geq 2$. If for every vertex x, $d^i(x) \geq d \geq 1$, $i \in \{1, 2, \cdots, c\}$, then G^c has a properly edge-colored path of length $2\lfloor\frac{c}{2}\rfloor d$.*

Proof. Identify all odd-numbered colors with color 1 and all the even-numbered ones with color 2. The resulting 2-edge-colored graph has minimum degree $\lfloor\frac{c}{2}\rfloor d$. Therefore, it has a properly edge-colored path of length $2\lfloor\frac{c}{2}\rfloor d$, as does the graph G^c. □

We believe that Corollary 2.2 is far from being best possible and may be improved. In fact, for given positive integers c and d, let G be a c-edge-colored graph on $cd+1$ vertices and such that each color class has degree d. Consider now a c-edge-colored graph G^c consisting of at least three copies of G having precisely one common vertex. Although the colored degree of G^c is d, it has no properly edge-colored path of length greater than $2cd$. Hence our conjecture:

Conjecture 2.3. *Let G^c be a c-edge colored graph, $c \geq 2$, such that for every vertex x, $d^i(x) \geq d \geq 1$, $i \in \{1, 2, \cdots, c\}$. Then G^c has a properly edge-colored path of length at least $min(n-1,\ 2cd)$.*

Let us now turn our attention to edge-colored complete regular graphs. The reader will recall that an edge-colored graph is regular if all its monochromatic spanning subgraphs are regular and of the same degree. Thus the order of such graphs is $cd+1$, where d is the degree of every monochromatic spanning subgraph and c is the number of colors used. Bollobás and Erdös in [8], conjectured that if the monochromatic degree of every vertex in K_n^c is strictly less than $\lfloor \frac{n}{2} \rfloor$, then K_n^c contains a properly edge-colored Hamiltonian cycle, for any $c \geq 3$. This conjecture was partially proved in [1] by using an advanced probabilistic method. The conjecture below is a weaker version of that conjecture by Bollobás and Erdös for regular edge-colored complete graphs, and perhaps an easier one to prove.

Conjecture 2.4. *Any c-edge-colored complete regular graph, $c \geq 3$, has a properly edge-colored hamiltonian cycle.*

Notice that this conjecture is not true for 2-edge-colored complete regular graphs, since such graphs have an odd number of vertices. Thus they may not contain properly 2-edge colored hamiltonian cycles. However, by Corollary 2.2, c-edge-colored complete regular graphs contain a properly edge-colored hamiltonian path for all even $c \geq 2$.

3 Multigraphs

This section is concerned with paths and cycles in edge-colored multigraphs. Notice that neither Corollary 2.2 nor Conjecture 2.3 extends to multigraphs since as many as n properly colored edges may occur between any pair of vertices in a perfect matching without there being any alternating hamiltonian cycle. However we have been able to prove Theorem 3.1 below. To be more specific in our statement of the theorem, we define a particular 2-edge-colored multigraph H_s as follows: Given an integer $s \geq 1$, consider an arbitrary tree on s vertices $t_1, t_2, ..., t_s$. Now replace each vertex t_i by a complete 2-edge colored multigraph T_i on $d+1$ vertices, for some even integer $d \geq 2$. For $s = 1$, define H_1 to be the graph T_1. Otherwise, for $s \geq 2$, H_s is obtained by assembling all T_i in such a way that T_i, T_j intersect in precisely one common point if and only if $t_i t_j$ is an edge of the tree. Clearly any longest properly edge-colored cycle in H_s has length d.

Theorem 3.1. *Let G^c be a c-edge colored multigraph, $c \geq 2$. Assume that for every vertex x, $d^i(x) \geq d \geq 1$, $i \in \{1, 2, \cdots, c\}$. Then G^c has either a properly-edge colored path of length at* *least $min\{n-1,\ 2d\}$ or else a properly-edge colored cycle of length $d+1$ unless G^c is isomorphic to H_1, in which case it has a cycle of length d.*

Proof. Let us suppose that the edges of G^c are colored with two colors, red and blue. Otherwise we may apply all arguments below to the spanning subgraph of G^c induced by the red/blue edges of G^c. Assume that G^c has no properly edge-colored path of length greater than or equal to $min\{n-1,\ 2d\}$, for otherwise we are finished. We shall show that G^c has a properly edge-colored cycle of length at least $d+1$ unless $G^c \cong H_1$. Let P denote a longest properly edge-colored path in G^c. By hypothesis, the length of P is at most $2d-1$, i.e, P has at most $2d$ vertices. Set $R = G^c - P$. Depending upon the parity of the length of P, set $P : x_1y_1x_2\cdots x_py_p$ or $P : x_1y_1x_2\cdots x_py_px_{p+1}$ for some integer $p \geq 1$. Assume without lost of generality that every edge x_iy_i, $1 \leq i \leq p$, is red while all other edges y_ix_{i+1} are blue. Observe that there is no vertex $z \in R$ such that the edge x_1z is blue, for otherwise the path $zx_1y_1x_2y_2\ldots$ will be longer than P, a contradiction to the choice of P. Similar arguments hold for the second endpoint of P. Consider now blue edges incident with x_1. Since the other endpoint of each such blue edge necessarily belongs to P, it follows that P has at least $d+1$ vertices.
Suppose first that the length of P is odd. Let us establish the following two facts.
Fact 1. For any blue edge y_ix_{i+1}, $1 \leq i \leq p-1$, of P, the edges x_1x_{i+1} and y_py_i (if any) cannot be both blue, otherwise the properly edge-colored cycle $x_1x_{i+1}y_{i+1}x_{i+2}\cdots y_py_ix_iy_{i-1}x_{i-2}\cdots x_1$ whould be of length greater than $2p \geq d+1$. Thus $d^b_{\{y_i,x_{i+1}\}}(x_1)+d^b_{\{y_i,x_{i+1}\}}(y_p) \leq 3$. Since there are $p-1$ blue edges on P, it follows that $\sum_{i=1}^{p-1}[d^b_{\{y_i,x_{i+1}\}}(x_1)+d^b_{\{y_i,x_{i+1}\}}(y_p)] \leq 3(p-1) = 3p-3$.
Fact 2. There are no blue edges x_1y_i, $\lceil\frac{d+1}{2}\rceil \leq i \leq p$, otherwise the cycle $x_1y_1\cdots y_ix_1$ whould be just as desired. Similarly, there are no blue edges x_iy_p, $1 \leq i \leq p-\lceil\frac{d+1}{2}\rceil+1$.
From Facts 1 and 2, $d^b_R(x_1) = d^b_R(y_p) = 0$ and the fact $p \leq d$, it follows that

$$\begin{aligned} d^b_P(x_1)+d^b_P(y_p) &\leq 3p-3-2\Big(p-\lceil\frac{d+1}{2}\rceil\Big) \\ &= p-3+2\lceil\frac{d+1}{2}\rceil \\ &= p-3+d+2, \\ &< 2d-1, \end{aligned}$$

a contradiction, since $d^b(x_1)+d^r(x_{p+1}) \geq 2d$.

Let us suppose now that the length of P is even. Consider first the case $2p+1 \geq d+2$. Observe, as in the foregoing, that :
1. If both edges x_1x_i, $x_{p+1}x_{i-1}$, $3 \leq i \leq p+1$ exist in G^c, then either x_1x_i is not blue or $x_{p+1}x_{i-1}$is not red.
2. There are no blue edges x_1y_i nor red edges $y_{p-i+1}x_{p+1}$, $\lceil\frac{d+1}{2}\rceil \leq i \leq p$.

As above, it follows that

$$\begin{aligned} d_P^b(x_1) + d_P^r(x_{p+1}) &\leq 3p - 3 - 2\Big(p - \lceil \frac{d+1}{2} \rceil\Big) \\ &= p - 3 + 2\lceil \frac{d+1}{2} \rceil \\ &\leq 2d - 1, \end{aligned}$$

again a contradiction, since $d^b(x_1) + d^r(x_{p+1}) \geq 2d$.

Suppose now that $2p+1 = d+1$. Since there is no blue edge x_1z, $z \in R$, and the minimum blue degree of x_1 is d, it follows that any edge x_1w, $w \in V(P) - \{x_1\}$ is blue. In particular, the edge x_1y_p is blue. Thus $C : x_1y_1 \cdots y_px_1$ is a properly edge-colored cycle of length d. Set $R' = G^c - C$ Assume first that R' is an independent set. Then any vertex z of R' is joined to any vertex of C with both blue and red parallel edges. If R' has at least two vertices, say z, z', then the path $zx_1y_px_p \cdots y_1z'$ is longer than P, a contradiction. If, on the other hand, R' is a singleton, then let z denote the unique vertex of R'. If $c = 2$, then G^c is isomorphic to H_1 and thus has a cycle of length d as claimed. Otherwise, if $c \geq 3$, consider an edge, say zx_i, $x_i \in V(C)$, in some color other than red/blue. Then the cycle, $zx_iy_{i-1}x_{i-1} \cdots y_iz$ has length $d+1$ as required.

It remains to consider the case where R' is not an independent set, i.e., R' has at least one edge, say xy. Choose xy with the property that either x or y, say x, is joined with an edge to at least one vertex, say w, of C (it is easy to verify that such vertices x, y, w exist in G^c). Observe that, if for some vertex w of C, $c(xw) \neq c(xy)$, then we can easily join xy to C in order to obtain a path longer than P, a contradiction to the maximality property of P. It follows that all edges between x and $V(C) \cup \{y\}$ are colored alike. Because of colored degree constraints, there must be some vertex z in R', distinct from y, such that $c(xz) \neq c(xy)$. Then, by appropriately concatenating the segment zxw within the cycle C we obtain again a path longer than P, a final contradiction. This completes the proof of the theorem. □

Theorem 3.1 above is partly improved upon in Theorem 3.6 given later, which deals with hamiltonian cycles in graphs with high colored degrees. Our aim now is at establishing a pair of lemmas with a view of proving Theorem 3.6.

Lemma 3.2. *Let G^c be a c-edge-colored multigraph on n vertices such that any vertex has minimum colored-degree greater than or equal to $\frac{n-1}{2}$. Then G^c has perfect matchings in any given color i, for n even, and an almost perfect matching for n odd.*

Proof. Choose any color, say red, and then consider a spanning subgraph G of G^c induced by the red edges of G^c. Clearly the minimum degree in G is at least $\frac{n-1}{2}$. By a well known theorem of Dirac [10], G has a hamiltonian path and therefore the conclusion of the lemma trivially holds. □

For a given color i, let M_i denote a matching of G^c in color i. The following definitions will be used in the sequel.

Definition 3.3. *A cycle $C : x_1x_2 \cdots x_{2s-1}x_{2s}x_1$ is compatible with M_i if either all edges $x_{2j}x_{2j+1}$ or all edges $x_{2j-1}x_{2j}$ belong to M_i, for any $j = 1, \cdots, s$ (all subscripts being modulo $2s$).*

Definition 3.4. *A path $P : p_1p_2 \cdots p_s$ is compatible with matching M_i if either the edges p_ip_{i+1}, $i = 1, 3, \cdots, s-1$ or the edges p_ip_{i+1}, $i = 2, 4, \cdots, s-2$ of P belong to M_i.*

The next two insightful lemmas will pave the way for the proof of the main theorem in this section.

Lemma 3.5. *Let G^c be a c-edge-colored multigraph, $c \geq 2$, with minimum colored degree $\lceil \frac{n}{2} \rceil$. Then G^c has a properly edge-colored cycle of length greater than or equal to $\lceil \frac{n}{2} \rceil + 1$ compatible with a maximum matching M_i of G^c for any fixed color $i \in \{1, 2, \cdots, c\}$.*

Proof. Let us suppose without loss of generality that the edges of G^c are colored with two colors (red/blue). Otherwise, instead of G^c, we may consider the spanning subgraph of G^c induced by its red/blue edges (since all arguments below apply to such spanning subgraphs). Let us fix a color, say red. Clearly G^c has a perfect red matching for n even and an almost perfect red matching for n odd, by Lemma 3.2. Let M_r denote this maximum red matching. Let now P: $p_1p_2 \dots p_t$ denote a path of maximum length compatible with M_r. We will prove this lemma by contradiction. For this, we assume that G^c has no properly edge-colored cycle of length greater than or equal to $\lceil \frac{n}{2} \rceil + 1$. We distinguish two cases depending on the parity of n. Let R denote the subgraph of G^c induced by $V(G^c) - V(P)$.

Case (a). n is even.

Assume first that the last edge of P is blue. As G^c has a red perfect matching, for some vertex z in R, the edge p_tz belongs to M_r. But then the path $p_1p_2 \dots p_tz$ is longer than P and compatible with M_r, a contradiction to the choice of P. It follows that both the first and last edges of P are colored red, and thus the length of P is odd. Furthermore, there is no blue edge p_tz for any $z \in V(R)$, otherwise the path $p_1p_2 \dots p_tz$ whould be compatible with M_c and longer than P, a contradiction to the choice of P. Consider now blue edges incident with p_1. Since the other endpoint of each such blue edge necessarily belongs to P, it follows that the number of vertices of P is at least $\lceil \frac{n}{2} \rceil + 1$, that is, $t \geq \lceil \frac{n}{2} \rceil + 1$.

Notice that for any blue edge p_ip_{i+1}, $i = 2, 4, \cdots, t-2$, of P, either $p_1p_{i+1} \in E^b$ or $p_tp_i \in E^b$ but not both, otherwise the properly edge-colored cycle $p_1p_{i+1}p_{i+2} \cdots p_tp_ip_{i-1}p_{i-2} \cdots p_1$ whould be of length greater than $\lceil \frac{n}{2} \rceil$. Here the number of blue edges on the path P is equal to $\frac{t}{2} - 1$. Set $\lceil \frac{n}{2} \rceil = 2r - 1$ or $\lceil \frac{n}{2} \rceil = 2r$, where r is a positive integer.

We distinguish now two subcases depending on the parity of $\lceil \frac{n}{2} \rceil$.

Subcase (a1): $\lceil \frac{n}{2} \rceil = 2r-1$ for some integer $r \geq 1$. None of the vertices $p_{2r}, p_{2r+2}, p_{2r+4}, \cdots, p_t$

is the other enpoint of any blue edge incident with p_1, otherwise we have a properly edge-colored cycle of length greater than $\lceil \frac{n}{2} \rceil$. Similarly, vertices $p_1, p_3, p_5, \cdots, p_{t-2r+1}$ are not the the other enpoints of blue edges incidents with p_t, otherwise we have a properly edge-colored cycle of length greater than $\lceil \frac{n}{2} \rceil$. So,

$$\begin{aligned} d_P^b(p_1) + d_P^b(p_t) &\leq 2(t-1) - \Big(\frac{t}{2} - 1\Big) - 2\Big(\frac{t}{2} - r + 1\Big) \\ &= \frac{t}{2} + 2r - 3. \end{aligned}$$

Observe also that $d_R^b(p_1) = d_R^b(p_t) = 0$. Thus

$$\begin{aligned} d^b(p_1) + d^b(p_t) &= d_P^b(p_1) + d_P^b(p_t) + d_R^b(p_1) + d_R^b(p_t) \\ &\leq \frac{t}{2} + 2r - 3 \\ &= \frac{t}{2} + \Big\lceil \frac{n}{2} \Big\rceil - 2 < n, \text{ which is impossible.} \end{aligned}$$

Subcase (a2). $\lceil \frac{n}{2} \rceil = 2r$. Vertices $p_{2r+2}, p_{2r+4}, p_{2r+6}, \cdots, p_t$ are not the other endpoints of blue edges incident with p_1, otherwise we have a properly edge-colored cycle of length greater than $\lceil \frac{n}{2} \rceil$. Similarly, vertices $p_1, p_3, p_5, \cdots, p_{t-2r-1}$ are not the other enpoints of blue edges incident with p_t, otherwise we have a properly edge-colored cycle of length greater than $\lceil \frac{n}{2} \rceil$. So we have

$$\begin{aligned} d_P^b(p_1) + d_P^b(p_t) &\leq 2(p-1) - \Big(\frac{t}{2} - 1\Big) - 2\Big(\frac{t}{2} - r\Big) \\ &= \frac{t}{2} + 2r - 1. \end{aligned}$$

As $d_R^b(p_1) = d_R^b(p_t) = 0$, we obtain,

$$\begin{aligned} d^b(p_1) + d^b(p_t) &= d_P^b(p_1) + d_P^b(p_t) + d_R^b(p_1) + d_R^b(p_t) \\ &\leq \frac{t}{2} + 2r - 1 \\ &= \frac{t}{2} + \Big\lceil \frac{n}{2} \Big\rceil - 1 < n, \text{ not possible.} \end{aligned}$$

Case (B). n is odd.

If $p_{t-1}p_t \in E^r$, our proof uses arguments very similar to those in Case (A). Assume therefore that $p_{t-1}p_t \notin E^r$. We have that the number of vertices of path P is greater than or equal to $\lceil \frac{n}{2} \rceil + 1$. Let us first see whether the number of vertices of path P is equal to $\lceil \frac{n}{2} \rceil + 1$ or not. If possible, let the number of vertices of path P be equal to $\lceil \frac{n}{2} \rceil + 1$, that is, $t = \lceil \frac{n}{2} \rceil + 1$. Then $p_{t-1}p_i \in E^b$, $i = 1, 2, \cdots, t$ ($\neq t-1$), otherwise a properly edge-colored path of length greater than $\lceil \frac{n}{2} \rceil + 1$ exists. Since the first edge p_1p_2 is red and the last edge $p_{t-1}p_t$ is blue, t must be odd and may be written as $t = 2q + 1$, where q is a positive integer. Now we consider one red edge $xy \in M_r$, x, $y \notin V(P)$. Since $d^b(x) \geq \lceil \frac{n}{2} \rceil$, vertex x is connected at least to one vertex of the path P with a blue edge in the form xp_{2i} or xp_{2i+1}. When $xp_{2i} \in E^b$, we have a properly edge-colored path $p_{2i+1}p_{2i+2}\cdots p_{t-1}p_1p_2\cdots p_{2i-1}p_{2i}xy$ of length greater than p. When $xp_{2i+1}\in E^b$, we have a properly edge-colored path $p_{2i}p_{2i-1}\cdots p_2p_1p_{t-1}p_{p-2}\cdots p_{2i+2}p_{2i+1}xy$ of length greater than p. So, our assumption is wrong and hence the number of vertices of path P is greater than or equal to $\lceil\frac{n}{2}\rceil+2$.

Now we delete the last blue edge from the path and find out the sum of the blue degrees of p_1 and p_{t-1} as in Case (A). Clearly, $d^b_R(p_1)=d^b_R(p_p)=0$.

Now when $\lceil\frac{n}{2}\rceil$ is odd, we have

$$\begin{aligned} d^b(p_1)+d^b(p_t) &= d^b_P(p_1)+d^b_P(p_pt)+d^b_R(p_1)+d^b_R(p_t)\\ &\leq \left\lfloor\frac{t}{2}\right\rfloor+\left\lceil\frac{n}{2}\right\rceil-2\\ &< n, \text{ not possible.}\end{aligned}$$

When $\lceil\frac{n}{2}\rceil$ is even, we have

$$\begin{aligned} d^b(p_1)+d^b(p_t) &= d^b_P(p_1)+d^b_P(p_t)+d^b_R(p_1)+d^b_R(p_t)\\ &\leq \left\lfloor\frac{t}{2}\right\rfloor+\left\lceil\frac{n}{2}\right\rceil-1\\ &< n, \text{ not possible.}\end{aligned}$$

Thus all the cases have been proved wrong, which validates our proof by contradiction. □

In the next theorem we prove degree conditions sufficient for an edge-colored multigraph to have a properly edge colored hamiltonian cycle. Our result may be viewed as a the counterpart to Dirac's well-known result for general graphs [10], insofar as the conditions involved deal only with degree conditions and nothing else.

Theorem 3.6. *Let G^c be a c-edge-colored multigraph of order n with minimum colored degree greater than or equal to $\lceil\frac{n+1}{2}\rceil$.*
I)If $c=2$, then G^c has a properly edge-colored hamiltonian cycle when n is even, and a properly edge-colored cycle of length $n-1$, when n is odd.
II) If $c\geq 3$, then G^c has a properly edge-colored hamiltonian cycle.

Proof of Case (I). By contradiction. First we assume that n is even. For a given color, say red, let us choose a maximum red matching M_r such that:
(1) A longest cycle $C: c_1c_2\cdots c_{m-1}c_mc_1$, compatible with M_r is the longest possible. By our hypothesis in connection with Lemma 3.5, we have $\frac{n}{2}+1\leq m\leq n-2$. In the sequel we will suppose that C is given such an orientation, so that edges c_ic_{i+1} are blue, for each even $i=2,\ 4,\cdots,(mod)\ m$. The remaining edges c_ic_{i+1} are red for each odd $i=1,\ 3,\cdots,(mod)\ m-1$.
(2) Among all maximum matchings obeying (1), consider a longest path $P: p_1p_2\cdots p_q$ of G^c-C compatible with M_r. Let R be the graph defined by $G^c-(C\cup P)$. Set $r=|R|$.

Since P is compatible with M_r, either the edges p_ip_{i+1}, $i=1,3,\cdots,q-1$ or the edges p_ip_{i+1}, $i=2,4,\cdots,q-2$ of P belong to M_r. We shall prove that, in fact, each edge p_ip_{i+1}, is in M_r

for every odd $i = 1,\ 3, \cdots, q-1$. To do so, it suffices to show by contradiction that both edges p_1p_2 and $p_{q-1}p_q$ belong to M_r. Suppose therefore that either p_1p_2 or $p_{q-1}p_q$, say $p_{q-1}p_q$ is not in M_r. Since vertex p_q is incident with some edge of M_r, there exists a vertex, say p_{q+1} such that $p_qp_{q+1} \in M_r$. Obviously, $p_{q+1} \in R$, since all vertices in $C \cup P$ are already incident with some edge of M_r. But then the path $p_1p_2 \cdots p_qp_{q+1}$ is longer than P, a contradiction to the maximality property of P. Hence the length of P is odd. Consider now edges colored in any color different from red, say blue, incident with p_1. We call the vertices of C are even and odd according to their position on the cycle. Definitely, vertex c_m on the cycle is an even one, as the edges of C alternate on red and blue colors. In order to facilitate discussion, a portion from an odd to an even vertex of C, say $c_{2l+1}c_{2l+2}c_{2l+3} \cdots c_{2t-1}\ c_{2t}$, will be called a segment if the following hold (here all indices are considered *modulo* m):
(i) both edges p_1c_{2l} and p_qc_{2t+1} are blue,
(ii) the edges p_1c_i and p_1c_j (if any) are not blue, for each even $i = 2l+2,\ 2l+2, \cdots,\ 2t$ and each odd $j = 2l+1,\ 2l+3, \cdots,\ 2t-1$.
With the definitions above, a segment has less vertices than P, otherwise $p_qc_{2t+1}c_{2t+2} \cdots c_{2l}p_1p_2 \cdots p_q$ is a cycle longer than C, a contradiction to the maximality property of C. In what follows we will distinguish between three cases depending upon the number of segments we may find on C. Namely: (a) There is no segment on C, (b) There is one segment on C and (c) There are more than one segments on C.
Case (a). There is no segment on C. Consider blue edges (if any) between p_1, p_q and C, say p_1c_i and p_qc_j. As there is no segment on C, either indices i and j are both even or both odd. Without loss of generality we can assume that are even. Let c_{2w} be a vertex of C such that either p_1c_{2w} or p_qc_{2w}, say p_qc_{2w}, is blue. If such a vertex c_{2w} does not exist on C, then

$$\begin{aligned} d^b(c_{2w+1}) + d^b(p_1) &= d^b_C(c_{2w+1}) + d^b_C(p_1) + d^b_P(c_{2w+1}) + d^b_P(p_1) + d^b_R(c_{2w+1}) + d^b_R(p_1) \\ &\leq 0 + 0 + 2(q-1) + 2r \\ &\leq n-4, \end{aligned}$$

a contradiction, since $m \geq \frac{n}{2} + 1$, thus $q + r \leq \frac{n-1}{2} - 1$.
Let us consider first blue edges (if any) between $\{p_1, c_{2w+1}\}$ and P. As the edge $c_{2w}p_q$ is a blue one, there is no blue edge $c_{2w+1}p_s$, for all odd $s = 1,\ 3, \cdots, q-1$. For otherwise, if $c_{2w+1}c_s \in E^b$, then the cycle $p_qc_{2w}c_{2w-1} \cdots c_{2w+2}c_{2w+1}p_sp_{s+1} \cdots p_q$ is longer than C a contradiction to the choice of C. Also, either $c_{2w+1}p_{s-1} \in E^b$ or $p_1p_s \in E^b$, but not both, otherwise, the cycle $p_qc_{2w}c_{2w-1} \cdots c_{2w+1}p_{s-1}p_{s-2} \cdots p_1p_sp_{s+1} \cdots p_q$ is longer than C. It follows that

$$d^b_P(c_{2w+1}) + d^b_P(p_1) \leq q.$$

Let us consider next blue edges between $\{p_1,\ c_{2w+1}\}$ and C. For each $z \neq w$, we have either $c_{2w+1}c_{2z+1} \in E^b$ or $p_1c_{2z} \in E^b$, but not both. Otherwise, the cycle $p_1c_{2z}c_{2z-1} \cdots c_{2w+1}c_{2z+1}c_{2z+2} \cdots c_{2w}p_qp_{q-1} \cdots p_1$ is longer than C. As p_1 is connected only to even vertices of C, from the above we obtain,

$$d^b_C(c_{2w+1}) + d^b_C(p_1) \leq m-1.$$

Recall also $d^b_R(p_1)=0$. It follows that

$$\begin{aligned} d^b(c_{2w+1}) + d^b(p_1) &= d^b_C(c_{2w+1}) + d^b_C(p_1) + d^b_P(c_{2w+1}) + d^b_P(p_1) + d^b_R(c_{2w+1}) + d^b_R(p_1) \\ &\leq m-1+q+r+0, \\ &\leq n-1, \end{aligned}$$

a contradiction.

Next we consider that there is at least one vertex on C, say c_{2w}, such that both p_1c_{2w} and p_qc_{2w} are blue edges. Recall also that $c_{2w+1}c_{2w}$ is a blue edge. In this case $d^b_p(c_{2w+1})=0$, for otherwise we may easily find a cycle longer than C. Also, similarly as before, we have $d^b_p(p_1) \leq q-1$ and $d^b_c(c_{2w+1}) + d^b_c(p_1) \leq m$. Now,

$$\begin{aligned} d^b(c_{2w+1}) + d^b(p_1) &= d^b_c(c_{2w+1}) + d^b_c(p_1) + d^b_p(c_{2w+1}) + d^b_p(p_1) + d^b_R(c_{2w+1}) + d^b_R(p_1) \\ &\leq m+q-1+r+0, \\ &\leq n-1, \end{aligned}$$

again a contradiction. This completes the proof of Case (a).

Case (b). There is precisely one segment on C. We let $S : c_{2l+1}c_{2l+2}\cdots c_{m-1}c_mc_1c_2\cdots c_{2t-1}c_{2t}$ denote the unique segment on C. Let s denote its length. Set $S^{'} = S \cup \{c_{2l}, c_{2t+1}\}$. Notice that the portion $C-S^{'}$ of C contains precisely $\frac{c-s-2}{2}$ edges. Notice also that the number of blue edges between $\{p_1,\ p_p\}$ and the endpoints of any blue edge in $C-S^{'}$ is at most two. In fact either p_1 and p_p are both connected to the same endpoint or one of p_1 and p_q (but not both) is connected to both endpoints of that blue edge. Otherwise we may easily define a properly edge-colored cycle with vertex set $V(P)\cup V(C)$. Thus of length greater than m. So we have

$$d^b_{C-S'}(p_1) + d^b_{C-S'}(p_q) \leq m-s-2.$$

Similarly for the vertices c_{2l+1} and c_{2t}, we get $d^b_{C-S'}(c_{2l+1}) + d^b_{C-S'}(c_{2t}) \leq m-s-2$. Also we have $d^b_R(c_{2l+1}) + d^b_R(c_{2t}) \leq 2(n-m-q)$. Recall also that $d^b_R(p_1) = d^b_R(p_q) = 0$.
Now,

$$\begin{aligned} d^b(p_1) + d^b(p_q) + d^b(c_{2l+1}) + d^b(c_{2t}) &= d^b_{C-S'}(p_1) + d^b_{C-S'}(p_q) + d^b_{C-S'}(c_{2l+1}) + d^b_{C-S'}(c_{2t}) \\ &\quad + d^b_{S'}(p_1) + d^b_{S'}(c_{2t}) + d^b_{S'}(p_q) + d^b_{S'}(c_{2l+1}) + d^b_P(p_1) \\ &\quad + d^b_P(c_{2t}) + d^b_P(p_q) + d^b_P(c_{2l+1}) + d^b_R(c_{2l+1}) + d^b_R(c_{2t}), \\ &\leq 2(n-s-q)-4+d^b_{S'}(p_1)+d^b_{S'}(c_{2t})+d^b_{S'}(p_q)+d^b_{S'}(c_{2l+1}) \\ &\quad + d^b_P(p_1) + d^b_P(c_{2t}) + d^b_P(p_q) + d^b_P(c_{2l+1}). \end{aligned} \tag{1}$$

Three subcases arises. Namely, (b1) $p_1c_{2l+1} \in E^b$ and $p_qc_{2t} \in E^b$, (b2) either $p_1c_{2l+1} \in E^b$ or $p_qc_{2t} \in E^b$ and (b3) neither $p_1c_{2l+1} \in E^b$ nor $p_qc_{2t} \in E^b$. Let us prove now these three subacases separately.

Subcase (b1). $p_1c_{2l+1} \in E^b$ and $p_qc_{2t} \in E^b$. In this subcase $p_1c_{2t+1} \notin E^b$, $p_qc_{2l} \notin E^b$, $c_{2l+1}c_{2t+1} \notin E^b$ and $c_{2t}c_{2l} \notin E^b$. Otherwise we may easily find a properly edge-colored cycle of length greater than m. Since $p_1c_{2l+1} \in E^b$ and $p_qc_{2t} \in E^b$, we have $p_1c_{2t} \notin E^b$, $p_1c_{2t+1} \notin E^b$, $p_pc_{2l} \notin E^b$ and $p_pc_{2l+1} \notin E^b$, otherwise a properly edge-colored cycle of length greater than m exists. Now we consider the blue edges between $\{p_1,\ c_{2t}\}$ and $S^{'}$. By the definition of segments, vertex p_1 can be connected to the vertices $c_{2l+1},\ c_{2l+3},\ c_{2l+5}, \cdots,\ c_{2t-5},\ c_{2t-3},\ c_{2t-1}$. However either $p_1c_k \in E^b$ or $c_{2t}c_{k-1} \in E^b$ for each $k = 2l+3, 2l+5, \cdots, 2t-3, 2t-1$, but not both. Otherwise the cycle $p_1, p_2, \cdots, p_q, c_{2t+1}c_{2t+2}\cdots c_{2l}c_{2l+1}c_{2l+2}\cdots c_{k-2}c_{k-1}c_{2t}\ c_{2t-1}\cdots c_kp_1$ has length greater than m, a contradiction to the choice of C. So, $d^b_{S'}(p_1) + d^b_{S'}(c_{2t}) \leq s+2$. Similarly, $d^b_{S'}(p_q)+d^b_{S'}(c_{2l+1}) \leq s+2$. Now we consider the blue edges between $\{p_1,\ c_{2t}\}$ and P. Vertex c_{2t} can not be connected with a blue edge to some of the vertices $p_2, p_4, \cdots, p_q$, for otherwise the cycle $p_qc_{2t+1}c_{2t+2}\cdots c_{2l}c_{2l+1}\cdots c_{2t-1}c_{2t}p_{2s+1}p_{2s+2}\cdots p_q$ (if $c_{2t}c_{2s+1} \in E^b$) has length greater than m. We also have either $c_{2t}p_k \in E^b$ or $p_1p_{k+1} \in E^b$, $k = 2, 4, \cdots, q-2$, but not both otherwise the properly edge-colored cycle $p_qc_{2t+1}c_{2t+2}\cdots c_{2l}c_{2l+1}\cdots c_{2t-1}c_{2t}p_kp_{k-1}\cdots p_1p_{k+1}p_{k+2}\cdots p_q$ has again length greater than m. So, $d^b_P(p_1)+d^b_P(c_{2t}) \leq q$. Similarly we have $d^b_P(p_q)+d^b_P(c_{2l+1}) \leq q$. Using these results in (1) we obtain $d^b(p_1) + d^b(p_q) + d^b(c_{2l+1}) + d^b(c_{2t}) \leq 2n < 2(n+1)$, a contradiction.

Subcase(b2). Either $p_1c_{2l+1} \in E^b$ or $p_qc_{2t} \in E^b$. Without loss of generality, we can assume that $p_1c_{2l+1} \in E^b$. In this subcase $p_qc_{2l} \notin E^b$, $p_qc_{2l+1} \notin E^b$, $p_1c_{2t} \notin E^b$ and $c_{2l}c_{2t} \notin E^b$. Otherwise we may easily define a properly edge-colored cycle of length greater than m. But it may be possible that $p_1c_{2t+1} \in E^b$ and $c_{2l+1}c_{2t+1} \in E^b$. Similarly as in Subcase (b1) we have $d^b_{S'}(p_1)+d^b_{S'}(c_{2t}) \leq s+3$, $d^b_{S'}(p_q)+d^b_{S'}(c_{2l+1}) \leq s+2$, $d^b_P(p_1)+d^b_P(c_{2t}) \leq q-1$, $d^b_P(p_q)+d^b_P(c_{2l+1}) \leq q$. Using these results in (1) we obtain $d^b(p_1) + d^b(p_q) + d^b(c_{2l+1}) + d^b(c_{2t}) \leq 2n < 2(n+1)$, not possible.

Subcase(b3). Neither $p_1c_{2l+1} \in E^b$ nor $p_qc_{2t} \in E^b$. In this subcase it may be possible that $p_1c_{2t+1} \in E^b$, $p_qc_{2l} \in E^b$, $c_{2l+1}c_{2t+1} \in E^b$ and $c_{2t}c_{2l} \in E^b$. Similarly as in Subcase (b1) we have $d^b_{S'}(p_1) + d^b_{S'}(c_{2t}) \leq s+3$, $d^b_{S'}(p_q) + d^b_{S'}(c_{2l+1}) \leq s+3$, $d^b_P(p_1) + d^b_P(c_{2t}) \leq q-1$ and $d^b_P(p_p) + d^b_P(c_{2l+1}) \leq q-1$. Using these results in (1) we obtain $d^b(p_1) + d^b(p_q) + d^b(c_{2l+1}) + d^b(c_{2t}) \leq 2n < 2(n+1)$, not possible. This completes the proof of Case (b)

Case (c). There are at least two distinct segments on C. Let $S_1 :\ \ c_{2l+1}c_{2l+2}\cdots c_{2t-1}c_{2t}$ and

$S_2 :\ c_{2z+1}c_{2z+2}\cdots c_{2w-1}c_{2w}$ be two distinct segments of C. Let s_1 and s_2 denote their lengths, respectively. Set $S_1^{'} = S_1 \cup \{c_{2l}, c_{2t+1}\}$ and $S_2^{'} = S_2 \cup \{c_{2z}, c_{2w+1}\}$. The number of blue edges on the portion $C - S_1^{'} - S_2^{'}$ on the cycle C is $\frac{m-s_1-s_2-4}{2}$. So we have $d^b_{C-S_1^{'}-S_2^{'}}(p_1)+d^b_{C-S_1^{'}-S_2^{'}}(p_q) \leq m-s_1-s_2-4$. Now, the number of blue edges on the segment S_1 is $\frac{s_1}{2}-1$ and the blue edges on the segment S_1 are $c_{2l+2}c_{2l+3}$, $c_{2l+4}c_{2l+5}, \cdots$, $c_{2l-4}c_{2l-3}$, $c_{2t-2}c_{2t-1}$. By the definition of segment, vertex p_1 may be connected with blue edges to the vertices $c_{2l+1}, c_{2l+3}, \cdots, c_{2t-3}c_{2t-1}$ on the segment S_1. Also vertex p_q may be connected with blue edges to the vertices $c_{2l+2}, c_{2l+4}, \cdots, c_{2t-2}c_{2t}$

on the segment S_1. For each blue edge c_kc_{k+1}, $k = 2l+2, 2l+4, \cdots, 2t-2$, on the segment S_1, either $p_1c_{k+1} \in E^b$ or $p_qc_k \in E^b$; but not both otherwise we have a properly edge-colored cycle $p_1c_{k+1}c_{k+2}\cdots c_{2t}c_{2t+1}c_{2t+2}\cdots c_{2l}c_{2l+1}\cdots c_{k-1}c_kp_qp_{q-1}\cdots p_1$ of length greater than m. Also we have either $p_1c_{2l+1} \in E^b$ or $p_qc_{2l} \in E^b$; but not both, otherwise a properly edge-colored cycle of length greater than m exists. Also we have either $p_qc_{2t} \in E^b$ or $p_1c_{2t+1} \in E^b$; but not both, otherwise a properly edge-colored cycle of length greater than m exists. Using these results we conclude that $d^b_{S_1'}(p_1) + d^b_{S_1'}(p_q) \leq \frac{s_1}{2} + 3$. Similarly for the segment S_2, we have $d^b_{S_2'}(p_1) + d^b_{S_2'}(p_q) \leq \frac{s_2}{2} + 3$. Moreover, we have $d^b_P(p_1) + d^b_P(p_q) \leq 2q-2$ and $d^b_R(p_1) = d^b_R(p_q) = 0$. Also $s_1 \geq q$ and $s_2 \geq q$. By considering the above inequalities we obtain,

$$\begin{aligned} d^b(p_1) + d^b(p_q) &= d^b_{S_1'}(p_1) + d^b_{S_1'}(p_q) + d^b_{S_2'}(p_1) + d^b_{S_2'}(p_p) + d^b_{C-S_1'-S_2'}(p_1) + d^b_{C-S_1'-S_2'}(p_q) \\ &\quad + d^b_P(p_1) + d^b_P(p_q) \\ &\leq m - \frac{s_1+s_2}{2} + 2q \\ &\leq m + q, \\ &\leq n, \end{aligned}$$

a contradiction.

Next we consider that n is odd. In this case we will show that graph G^c has properly edge-colored cycle of order $n-1$. Let C, P and R be defined as for n even. Assume by contradiction that C has length m, $\lceil \frac{n}{2} \rceil + 1 \leq m \leq n-3$. If both edges p_1p_2 and $p_{q-1}p_q$ have a same color, say red, then we may complete the argument, as for n even. Assume therefore that edges p_1p_2 and $p_{q-1}p_q$ have different colors, say $c(p_1p_2)$ is red and $c(p_{q-1}p_q)$ is blue. Now, we can complete the proof by considering the path $P' : p_1p_2\cdots p_{q-1}$ instead of P and then apply again all arguments used for n even. Hence the proof of Case (I) □

Proof of (II). Let us consider the spanning subgraph H of G^c induced by all edges on two distinct colors, say red and blue, i.e.,$V(H) = V(G^c)$ and $E(H) = E^r(G^c) \cup E^b(G^c)$. If n is even, then H has a properly edge-colored red/blue hamiltonian cycle, thus the conclusion follows for G^c. Assume therefore that n is odd. Again, by Case (I), there exists some vertex z in H such that $H - z$ has a properly edge-colored red-blue cycle, say $C : x_1y_1\cdots x_{\frac{n-1}{2}}y_{\frac{n-1}{2}}$ spanning the $n-1$ vertices of $H - z$. Suppose that all edges x_iy_i (modulo $\frac{n-1}{2}$) are red, while all other edges y_ix_i (modulo $\frac{n-1}{2}$) of C are blue. Pick now any red edge x_iy_i. Assume first that the number of red and , say green (i.e., any third color not used on the cycle) edges between $\{x_i, y_i\}$ and z is greater than or equal to 3. Then either the edge zx_i is red and the edge zy_i is green or zx_i is green and th e edge zy_i is red. But either the cycle $x_1y_1\cdots x_izy_i\cdots x_{\frac{n-1}{2}}y_{\frac{n-1}{2}}x_1$ or the cycle $y_1x_1\cdots y_izx_i\cdots y_{\frac{n-1}{2}}x_{\frac{n-1}{2}}y_1$ is a properly edge-colored hamiltonian one. Assume therefore that the number of red and green edges is less than or equal to two. Since there are $\frac{n-1}{2}$ red edges on C, it follows that $d^r(z) + d^g(z) \leq 2\frac{n-1}{2} = n-1$, a contradiction since $d^r(z) + d^g(z) \geq n+1$. Hence the theorem. □

Notice that the conditions of previous theorem are not far from being the best possible. Indeed, let k and c be two arbitrary integers, $k \geq 1, c \geq 2$. Consider a multigraph on $2k+1$ vertices, consisting of two c-edge-colored complete multigraphs each of order $k+1$, having precisely one common vertex. Such a graph has no hamiltonian cycle although its minimum colored degree is $k(=\frac{n-1}{2})$. In fact, we believe that the following is true.

Conjecture 3.7. *Statement of Theorem 3.6 remains true, if we replace* $\lceil \frac{n+1}{2} \rceil$ *by* $\lceil \frac{n}{2} \rceil$.

From Theorem 3.6 we obtain a series of corollaries for properly edge-colored hamiltonian paths.

Corollary 3.8. *Let* G^c *be a c-edge colored multigraph,* $c \geq 3$. *Assume that for each color* $i \in \{1, 2, \cdots, c\}$ *and for each vertex* x *of* G^c, $d^i(x) \geq \lceil \frac{n}{2} \rceil$. *Then* G^c *has a properly edge-colored hamiltonian path.*

Proof. Consider a new graph H obtained from G^c by adding a new vertex x and all possible edges between x and G^c for each color $i \in \{1, 2, \cdots, c\}$. Now it is not difficult to see that H satisfies all conditions of Theorem 3.6 and therefore it contains a properly edge-colored hamiltonian cycle. Now a hamiltonian path in G^c may be obtained by removing x from that hamiltonian cycle of H. $\square$

The conditions of previous corollary are not far from being best possible. This may be shown by a multigraph on $2k$ vertices, consisting of two c-edge-colored complete multigraphs each of order k, without common vertices. Such a graph has no hamiltonian cycle although its minimum colored degree is $k-1(=\frac{n-2}{2})$.
In next corollary we are interested for properly edge-colored hamiltonian paths with fixed endpoints.

Corollary 3.9. *Let* x, y *be two fixed vertices in* G^c, $c \geq 2$. *Assume that* $\forall v \in V(G^c)$, $d^i(v) \geq \lceil \frac{n+3}{2} \rceil$ *for each color* $i \in \{1, 2, \cdots, c\}$. *Then* G^c *has a properly edge-colored hamiltonian path with endpoints* x, y.

Proof. Assume first n is odd. Let H be a new 2-edge colored multigraph, on two colors red and blue, obtained from G^c as follows. Concatenate x, y to a new vertex z in H, i.e., $V(H) = V(G^c) - \{x, y\} \cup \{z\}$. In addition, for each vertex w in $V(G^c) - \{x, y\}$ add the edge wz in H if the edge wx (respectively wy) is red (respectively blue) in G^c. Finally delete all edges in the subgraph induced by $V(G^c) - \{x, y\}$ which are on colors other than red and blue. Now it is not difficult to see that H has $n-1$ vertices and its minimum colored degree is greater than or equal to $\lceil \frac{n+3}{2} \rceil - 1 = \lceil \frac{(n-1)+1}{2} \rceil$. Thus it satisfies all conditions of Theorem 3.6 and therefore it contains a properly edge-colored hamiltonian cycle. Now a hamiltonian path between x and

y in G^c may be obtained by deleting z on that hamiltonian cycle of H and replacing it by x, y. Assume next that n is even. Let now H be a new 2-edge-colored multigraph obtained from G^c by deleting vertices x, y. Delete also all edges in $G^c - x - y$ which are on colors other than red and blue. Then the order of H is $n-2$ and its minimum colored degree is $\lceil \frac{n+3}{2} \rceil - 1 = \frac{n}{2} = \frac{n-2}{2} + 1 = \lceil \frac{(n-2)+1}{2} \rceil - 1$. Thus H has a properly 2-edge-colored hamiltonian cycle C. Set $C : x_1 y_1 \cdots x_p y_p x_1$, where $p = \frac{n-2}{2}$. Assume without loss of generality that all edges $x_i y_i$ (modulo p) are red, while the remaining edges $y_i x_i$ (modulo p) of C are blue. Pick a red edge $x_i y_i$ and then observe that if the edge $x x_i$ (if any) is red, then the edge $y y_i$ (if any) is not red. Otherwise the path $x x_i y_{i-1} \cdots y_i y$ should be the desired one. Similarly one of the two edges $y x_i$ or $x y_i$ (if any) is not red. Thus, $d^r_{\{x_i,y_i\}}(x) + d^r_{\{x_i,y_i\}}(y) \leq 2$. Since there are p such red edges $x_i y_i$ on C, it follows that $d^r_C(x) + d^r_C(y) = \sum_{i=1}^{p\ (modulo\ p)} d^r_{\{x_i,y_i\}}(x) + d^r_{\{x_i,y_i\}}(y) \leq 2p \leq 2\frac{n-2}{2} = n-2$. However this is a contradiction, since $d^r_C(x) + d^r_C(y) = d^r(x) + d^r(y) - d^r_y(x) - d^r_x(y) \geq 2\lceil \frac{n+3}{2} \rceil - 2 = n+2$. □

The conditions of previous corollary are not far from being best possible. This may be shown by a multigraph on $2k+2$ vertices, consisting of two c-edge-colored complete multigraphs each of order $k+2$, with precisely two common vertices x and y. Such a graph has no properly edge-colored hamiltonian path with extremities x, y, although its minimum colored degree is $k+1$. Also, it should be interesting to study the questions of the above corollary in the case of edge-colored complete graphs. In particular, the following problems seem interesting.

Problem 3.10. *[6] Let x, y be two given vertices in a c-edge colored complete (multi)graph K^c_n, $c \geq 2$. Is there any polynomial algorithm for finding, if any, a properly edge-colored hamiltonian path between x, y in K^c_n?*

Problem 3.11. *Let x be a given vertex in a c-edge colored complete (multi)graph K^c_n, $c \geq 3$. Is there any polynomial algorithm for finding, if any, a properly edge-colored hamiltonian path starting from x in K^c_n such that the color of the first edge of this path is fixed ?*

Problem 3.12. *Let x, y be two given vertices in a c-edge colored complete (multi)graph K^c_n, $c \geq 2$. Is there any polynomial algorithm for finding, if any, a properly edge-colored hamiltonian path between x, y in K^c_n such that the colors of the first or last edge (or of both first and last edges) of this path are fixed ?*

In view of Theorem 3.14 below we establish the following lemma which could be of independent interest.

Lemma 3.13. *Let G^c be a c-edge colored multigraph, $c \geq 2$. Assume that G^c contains a properly edge-colored cycle C on two colors red and blue of length $2p < n$. Assume furthermore that there exists a vertex x in $G^c - C$ such that (red/blue degrees) $d^r_C(x) > p$ and $d^b_C(x) > p$ for $c = 2$, or (red/green degrees) $d^r_C(x) > p$ and $d^g_C(x) > p$, for $c \geq 3$.*
i)If $c = 2$, then G^c has properly edge-colored cycles of all even lengths $2, 4, \cdots, 2p$ through x.
ii)If $c \geq 3$, then G^c has properly edge-colored cycles of all lengths $2, 3, 4, \cdots, 2p+1$ through x.

Proof. Set $C: x_1y_1\cdots x_py_px_1$. Assume without loss of generality that all edges x_iy_i (modulo p) are red, while the remaining edges y_ix_i (modulo p) of C are blue. Define $X = \{x_i | x_i \in V(C) i = 1, 2, \cdots, p\}$ and $Y = \{y_i | y_i \in V(C) i = 1, 2, \cdots, p\}$. For two given colors $s, t \in \{r, b, g\}$, consider the degree-sum $d^s_C(x)+d^t_C(x)$ and rewrite it as $d^s_X(x)+d^s_Y(x)+d^t_X(x)+d^t_Y(x) = d^s_X(x) + d^t_X(x) + d^s_Y(x) + d^t_Y(x)$. By definition,

$$d^s_X(x) + d^t_X(x) + d^s_Y(x) + d^t_Y(x) > 2p \ (*) \ .$$

From (*), it follows that, either $d^s_X(x) + d^t_X(x) > p$ or $d^s_Y(x) + d^t_Y(x) > p$. Assume without loss of generality that

$$d^s_X(x) + d^t_X(x) > p \ (**)$$

Now we are ready to prove Cases (i) and (ii).

Proof of (i): Consider (**), by setting $s = r$ (red) and $t = b$ (blue). Thus $d^r_X(x) + d^b_X(x) > p$. Assume now by contradiction that for some even k, $2 \leq k \leq 2p$, there exist no properly edge-colored cycle of length k through x in G^c. This means that for any $i = 1, 2, \cdots, p$ (modulo p), going clockwise on the cycle, if the edge x_ix (if any) is blue, then the edge $x_{i+\frac{k}{2}-1}x$ (if any) is not red. Otherwise the cycle $xx_iy_i\cdots x_{\frac{k}{2}-1}x$ should be properly edge-colored and of even length k, a contradiction to our assumption. Thus, $d^b_{x_i}(x) + d^r_{x_{\frac{k}{2}-1}}(x) = 0$ or 1. It follows that $p < d^b_X(x) + d^r_X(x) = \Sigma_{i=1}^{p \ (modulo \ p)} d^b_{x_i}(x) + d^r_{x_{\frac{k}{2}-1}}(x) \leq p$, a contradiction. This completes the argument and the proof of this case.

Proof of (ii): Assume now by contradiction that for some integer k, $2 \leq k \leq 2p + 1$, there exist no properly edge colored cycle of length k through x in G^c. If k is even, then, complete the argument by using arguments similar to those of Case (i). For k odd, it follows from (*) that, either $d^r_X(x) + d^g_Y(x) > p$ or $d^r_Y(x) + d^g_X(x) > p$. Assume without loss of generality that $d^r_X(x) + d^g_Y(x) > p$. Going anti-clockwise on the cycle, observe that for any $i = p, p-1, \cdots, 2, 1$ (modulo p), if the edge x_ix (if any) is blue, then the edge $y_{i-\lfloor\frac{k}{2}\rfloor}x$ (if any) is not red. Otherwise the cycle $xx_iy_{i-1}\cdots x_{i+1-\lfloor\frac{k}{2}\rfloor}y_{i-\lfloor\frac{k}{2}\rfloor}x$ should be a properly edge-colored one of length k, a contradiction to our assumption. Thus, $d^r_{x_i}(x) + d^g_{y_{i-\lfloor\frac{k}{2}\rfloor}}(x) = 0$ or 1. It follows that $p < d^r_X(x) + d^g_Y(x) = \Sigma_{i=1}^{p \ (modulo \ p)} d^r_{x_i}(x) + d^g_{y_{i-\lfloor\frac{k}{2}\rfloor}}(x) \leq p$, a contradiction. This completes the argument and the proof of this lemma. □

In next theorem, we go further by showing that under the conditions of Theorem 3.6, G^c has cycles of many lengths.

Theorem 3.14. *Let G^c be a c-edge colored multigraph, $c \geq 2$. Assume that $\forall x \in V(G^c)$, $d^i(x) \geq \lceil\frac{n+1}{2}\rceil$ for each color $i \in \{1, 2, \cdots, c\}$.*
i)If $c = 2$, then G^c is even-pancyclic.
ii)If $c \geq 3$, then G^c is pancyclic.

Proof. Assume first that n is odd. Using Theorem 3.6 we conclude that G^c has a properly edge-colored cycle C of length $n-1$ with two colors, say red and blue. Let x be the vertex G^c not included in C. Now, by considering x and C, it is an easy exercise to see that all conditions of Lemma 3.13 are full filled, so the conclusion follows.
Assume next that n is even. Pick any vertex x and consider the graph $H \cong G^c - x$. Since n is even, the order of H, i.e. the number $n-1$, is odd. Furthermore the minimum colored degree of any vertex in H is greater than or equal to $\lceil \frac{n+1}{2} \rceil - 1 = \frac{n}{2} + 1 - 1 = \frac{n}{2} = \frac{(n-1)+1}{2} = \lceil \frac{(n-1)+1}{2} \rceil$. It follows from Theorem 3.6 that H has a properly-edge colored cycle C of length $n-2$. Using this fact and Lemma 3.13 we complete the argument, since any i-colored degree of x on C satisfies $d^i_C(x) \geq \lceil \frac{n+1}{2} \rceil - 1 \geq \frac{n}{2} > \frac{n-2}{2}$. Hence the theorem. □

Notice that degree conditions of Theorem 3.14 cannot be relaxed as shown by the edge-colored complete bipartite multigraph $K^c_{\frac{n}{2},\frac{n}{2}}$. Although, such a graph has minimum colored degrees $\frac{n}{2}$, it has no cycles of odd length. Notice also that case $c = 2$ cannot be considered for pancyclicity, since a 2-edge-colored graph has no properly edge-colored cycles of odd length.
We conclude this section with the following result on edge-colored random multigraphs.

Theorem 3.15. *Let C denote a sufficiently large constant and let $\mathcal{G}^{(b)} = \mathcal{G}^{(b)}(2n,p)$, $\mathcal{G}^{(r)} = \mathcal{G}^{(r)}(2n,p)$ be two independent random graphs on the same vertex set $V = \{1,2,...,2n\}$ and with the same edge probability $p = Cn^{-1}\log n$. The edges of $\mathcal{G}^{(b)}$ are colored blue, and the edges of $\mathcal{G}^{(r)}$ are colored red. Then, with probability tending to 1, as n tends to infinity, the random multigraph $\mathcal{G} = \mathcal{G}^{(b)} \cup \mathcal{G}^{(r)}$ has a properly edge-colored hamiltonian cycle.*

Proof. By a known result (see Theorem 24, page 167 in [7]), for sufficiently large C, the graph $\mathcal{G}^{(b)}$ has a perfect matching with probability tending to 1. Let such a perfect matching be $\{\{2k+1, 2k+2\} : 0 \leq k \leq n-1\}$. We will prove that the $\mathcal{G}$ contains a properly edge-colored hamiltonian cycle which uses all the edges $\{2k+1, 2k+2\}$ in the direction $2k+1 \rightarrow 2k+2$. Clearly the remaining edges of that cycle will be red. For this, let us consider the directed graph $D = (V(D), A(D))$ with vertex set $V(D) = \{v_1, v_2, ..v_n\}$ and $A(D)$ defined as follows: For each $0 \leq k < \ell \leq n-1$, we do the following:
- The arc $(v_k, v_\ell) \in A(D)$ if and only if the edge $\{2k+2, 2\ell\} \in \mathcal{G}^{(r)}$.
- The arc $(v_\ell, v_k) \in A(D)$ if and only if the edge $\{2k+1, 2\ell+2\} \in \mathcal{G}^{(r)}$. Note that, conditionally

on the pairing, D is independent of $\mathcal{G}^{(b)}$. Now assume that D has an hamiltonian circuit, say $v_{i_1}, v_{i_2}, ..., v_{i_n}$. Clearly, replacing each vertex v_{i_j} by the edge $(2i_j+1, 2i_j+2)$ gives an hamiltonian properly edge-colored cycle of $\mathcal{G}$. Thus we are only left with the task of asserting that D has an hamiltonian circuit, with high probability. But the arc probabilities in D are exactly the edge probabilities in $\mathcal{G}^{(r)}$. Again, it is a standard result that a random directed graph with arc probabilities $Cn^{-1}\log n$ has a hamiltonian circuit with high probability for large C [2]. Thus the assertion is true and the theorem is proved. □

4 Conclusion

This paper is the result of persistent effort at systematising results on properly edge-colored paths and cycles by drawing upon anologous theorems from uncolored graphs. Although some notion of connectivity in edge-colored graphs have already been known (see, e.g. Chapter 11 in [3]), in the absence of a counterpart to Menger's theorem and network flow theory, the task may seem daunting at first, perhaps even beyond reach. Yet the results are surprinsingly consistent with their counterparts from Graph Theory. It seems as though we can phrase the same theorems in their properly coloring versions and get valid theorems, except for the notable fact that the proofs are sometimes long and tedious and the work to get them is fraught with intricacies and unsuspected difficulties. In the final analysis, it is remarkable, in our view, that such a theorem as Dirac's [10], should carry over, almost word for word, to the case mentioned in this paper. The proofs, albeit difficult, rely on little more than the pigeonhole principle and, in some important instances, on matching theory and related subjects.

The paper is sprinkled throughout with numerous conjectures of our own devising, which bears witness to the liveliness of this line of research. On the subject of conjectures, we would like to record our recognition of the contribution of both referees for promptly (almost on the fly, as it were) pointing out that two of the conjectures that appeared in an earlier version of this paper were false, as they pointed to counter-examples to that effect. In fairness to the unknown referees then, we present the two conjectures along with some references to the counter-examples that served to disprove them:

Conjecture 4.1. *Let G^c be a c-edge colored graph, $c \geq 2$, such that for every vertex x, $d^i(x) \geq d \geq 1$, $i \in \{1, 2, \cdots, c\}$.*
i) If $c = 2$, G^c has a properly edge-colored cycle of length $2d$.
ii)If $c \geq 3$, G^c has a properly edge-colored cycle of length $cd + 1$.

Conjecture 4.2. *Every c-edge-colored multigraph G^c, $c \geq 2$, with minimum colored degree d has :*
i) a properly edge-colored cycle of length $d + 1$ unless $c = 2$ and $G^c \cong H_s$ in which case G^c has a cycle of length d and
ii) a properly edge-colored path of length $min\{n - 1,\ 2d\}$

Published counter-examples are found in [12], where it is proved that there exist edge-colored graphs with minimum colored degrees d and without properly edge-colored cycles.

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