Scaled Dimension and the Berman-Hartmanis Conjecture

Mathias Hauptmann *

Abstract

In 1977, L. Berman and J. Hartmanis [BH77] conjectured that all polynomial-time many-one complete sets for NP are are pairwise polynomially isomorphic. It was stated as an open problem in [LM99] to resolve this conjecture under the measure hypothesis from quantitative complexity theory. In this paper we study the polynomial-time isomorphism degrees within $\deg_m^p(SAT)$ in the context of polynomial scaled dimension. Our results are the following:

1. We consider scaled dimensions of order in between -2 and -3. Especially we define scaled dimensions $\dim_p^{(-2,k)}, k \in \mathbb{N}$.
2. Let $ISO_m^p(SAT)$ denote the polynomial-time isomorphism degree of SAT. While for each k, $\dim_p^{(-2,k)}(\deg_m^p(SAT)) = \dim_p^{(-2,k)}(NP)$, if r is a growth rate function of order smaller than every order $(-2,k)$, then $\dim_p^{(r)}(ISO_m^p(SAT)) = 0$.
3. We consider the class of disjoint unions $L_1 \oplus L_2$ of NP-complete languages L_1, L_2 such that L_1 and L_2 are polynomially isomorphic. We show that for $|i| \leq 2$ the i-th order scaled dimension of this class equals that of NP. The same holds for the scaled dimensions $\dim_p^{(-2,k)}$.

1 Introduction

The Berman-Hartmanis Conjecture, also known as the Isomorphism Conjecture, states that all NP-complete languages are pairwise polynomially isomorphic [BH77], where two sets A and B are polynomially isomorphic iff there exists a bijective polynomial-time computable, polynomial-time invertible reduction from A to B. The Isomorphism Conjecture is unsolved even under assumption $P \neq NP$. Kurtz et al. [KMR89] proved the failure of the Isomorphism Conjecture relative to a random oracle, and Rogers [Rog95] was able to construct an oracle relative to which the conjecture holds and one-way functions exist.

Lutz [Lut00] defined polynomial dimension as a generalization of Hausdorff dimension and gave a characterization of polynomial dimension in terms of efficiently computable

*Dept. of Computer Science, University of Bonn, Email: hauptman@cs.uni-bonn.de

gales. While small-span theorems were proved for polynomial measure [JL95], Ambos-Spies et al. [ASMRS01] proved the impossibility of small span theorems for polynomial dimension. Scaled dimension was introduced by Hitchcock, Lutz and Mayordomo [HLM] in order to overcome certain limitations and difficulties of polynomial dimesion. Hitchcock [Hit04] gives several results concerning scaled dimensions of lower and upper spans and of polynomial-time degrees. Especially he shows that for $|i| \leq 2$, the i-th order scaled dimension of the polynomial-time m-degree of SAT equals that of NP, while for $i = -3$ the i-th order scaled dimension of this degree vanishes. It was stated in [LM99] as an open problem to resolve the Isomorphism Conjecture under reasonable hypotheses from Quantitative Complexity Theory.

In this paper we study scaled dimensions of polynomial-time isomorphism degrees inside NP. We give the following refinement of the scaled-dimension results obtained in [Hit04]: We define a class of growth rates $r_{-2,k}, k \in \mathbb{N}$ of order in between -3 and -2. It turns out that for each k, the associated scaled dimension of the polytime m-degree of SAT equals that of NP, while for each growth rate r that is in a certain sense of lower order than all the $r_{-2,k}$, the scaled dimesion of the polynomial-time isomorphism degree of SAT vanishes.

Furthermore we consider classes of disjoint unions $L_1 \oplus L_2$ of languages L_1, L_2 from NP. We prove that for each $|i| \leq 2$ as well as for each growth rate $r_{-2,k}$, the according scaled dimension of the set of all $L_1 \oplus L_2$ such that L_1, L_2 are in the polytime m-degree of SAT and such that L_1 and L_2 are polynomial-time isomorphic equals the scaled dimension of NP, while for scales order lower than that of every $r_{-2,k}$,the scaled dimension of this class is equal to 0. This result is obtained by combining techniques from [Hit04] with a new operator that transforms a disjoint union $L_1 \oplus L_2$ into a disjoint union of two polynomial-time isomorphic sets.

The rest of the paper is organized as follows: In section 2 we give provide some standard notions and definitions concerning p-dimension and scaled dimensions. In section 3 we introduce the new class of growth functions of order between -3 and -2. In section 4 we consider scaled dimensions of the polynomial-time isomorphism degree of SAT, and in section 5 we study classes of disjoint unions of languages from NP.

2 Preliminaries

Let $\leq_m^p$ denote polynomial-time many-one reducibility. Given a set $A \subseteq \{0,1\}^*$, let $P_m(A)$ and $P_m^{-1}(A)$ denote the lower cone and upper cone of A with respect to polynomial-time many-one reducibility, i.e. $P_m(A) = \{B \subseteq \{0,1\}^* | B \leq_m^p A\}$, $P_m^{-1}(A) = \{B \subseteq \{0,1\}^* | A \leq_m^p B\}$. The polynomial-time m-degree of A is $\deg_m^p(A) = P_m(A) \cap P_m^{-1}(A)$. Two sets $A, B \subseteq \{0,1\}^*$ are *polynomially isomorphic* (denoted as $A \simeq_p B$) iff there exists a bijection $f \colon \{0,1\}^* \to \{0,1\}^*$ such that both f and f^{-1} are polynomial-time computable and f is a many-one reduction from A to B. Let $ISO_m^p(A)$ denote the polynomial-time isomorphism degree of A, i.e. $ISO_m^p(A) = \{B \subseteq \{0,1\}^* | A \simeq_p B\}$.

Resource-bounded measure and resource-bounded dimension [Lut00] are used to study the quantitavive structure of complexity classes. Hitchcock, Lutz and Mayordomo [HLM] defined scaled dimension which was subsequently used in order to study quantitative properties of complexity classes (see for instance [Hit04], [HLVM], [HPV]).

A martingale is a function $d\colon \{0,1\}^* \to [0,\infty)$ such that for each $x \in \{0,1\}^*$, $d(x0) + d(x1) = 2 \cdot d(x)$. A martingale d succeeds on a language $L \subseteq \{0,1\}^*$ iff $\limsup_{n\to\infty} d(A|n) = \infty$. $S^\infty[d]$ is the set of all languages $L \subseteq \{0,1\}^*$ on which d succeeeds. Let $\mathcal{C} \subseteq 2^{\{0,1\}^*}$ be a class of languages. $\mathcal{C}$ has p-measure zero ($\mu_p(\mathcal{C}) = 0$) iff there is some polynomial-time computable martingale d such that $\mathcal{C} \subseteq S^\infty(\mathcal{C})$. As a generalization of Hausdorff dimension, p-dimension was characterized in terms of the growth rate of martingales. The p-dimension of a class $\mathcal{C} \subseteq 2^{\{0,1\}^*}$ is defined the infimum of all $s > 0$ such that there is some polytime-computable martingale d such that for all $A \in \mathcal{C}\ \exists^\infty n\ d(A|n) \geq (1-s) \cdot n$. Lutz et al. [HLM] defined $g(n,s)$-scaled dimension by replacing martingales by g-scaled s-supergales in the above definition. A function $d\colon \{0,1\}^* \to [0,\infty)$ is called g-scaled s-supergale iff for each $x \in \{0,1\}^*$, $d(x) \geq 2^{-\Delta g(|x|,s)}(d(x0) + d(x1)$, where $\Delta g(m,s) = g(m+1,s) - g(m,s)$. Equivalently scaled dimension can be defined in terms of *growth rate functions*. A function $r\colon H \times [0,\infty) \to \mathbb{R}$, $H = (a,\infty)$ for some $a \in \mathbb{R} \cup \{-\infty\}$ is called a *growth rate function* iff the following conditions hold:

(1) There exist constants c, c' such that for each $n \in H$, $r(n,1) = c$, $r(n,0) = n + c'$.

(2) For n sufficiently large, $s \mapsto r(n,s)$ is strictly decreasing.

(3) For all $s' > s \geq 0$, $\lim_{n\to\infty}[r(n,s') - r(n,s)] = -\infty$.

Especially, in [HLM] for each integer i, a scale g_i was defined, and $\dim_p^{(i)}$ denotes the i-th order scaled dimension. The associated growth rate function is given by $r_i(n,s) = n + 2^{-(i+1)} - g_i(n,s)$ for $i < 0$ and $r_i(n,s) = n - g_i(n,s)$ for $i \geq 0$. The following table lists the scales g_i and associated growth rate functions r_i for $|i| \leq 3$.

Scale	Growth Rate Function
$g_{-3}(n,s) = n + 4 - 2^{2^{(\log\log n)^{1-s}}}$	$r_{-3}(n,s) = 2^{2^{(\log\log n)^{1-s}}}$
$g_{-2}(n,s) = n + 2 - 2^{(\log(n))^{1-s}}$	$r_{-2}(n,s) = 2^{(\log(n))^{1-s}}$
$g_{-1}(n,s) = n + 1 - n^{1-s}$	$r_{-1}(n,s) = n^{1-s}$
$g_0(n,s) = s \cdot n$	$r_0(n,s) = (1-s) \cdot n$
$g_1(n,s) = n^s$	$r_1(n,s) = n - n^s$
$g_2(n,s) = 2^{(\log n)^s}$	$r_2(n,s) = n - 2^{(\log n)^s}$
$g_3(n,s) = 2^{2^{(\log\log n)^s}}$	$r_3(n,s) = n - 2^{2^{(\log\log n)^s}}$

3 Growth Rates of Order between -2 and -3

For each non-negative integer k we define $r_{-2,k}(n,s) = 2^{(\log(n))^{\frac{1-s}{k^s}}}$. Note that $r_{-2,1} = r_{-2}$. The next lemma states that the functions $r_{-2,k}$ are growth rates of order in between -2 and -3.

Lemma 3.1. *For each $k \geq 1$, $r_{-2,k}$ is a growth rate function. Furthermore, for each $s \in (0,1)$ $r_{-2,k+1}(n,s) = o(r_{-2,k}(n,s))$, $r_{-2,k}(n,s) = \omega(r_{-3}(n,s))$.*

Proof. From the definition of functions $r_{-2,k}$ we obtain $r_{-2,k}(n,1) = 2, r_{-2,k}(n,0) = n$. Since for each $k \geq 1$, the exponent $\frac{1-s}{k^s}$ is monotone decreasing in s, property (3) in the definition of growth rate functions is satisfied as well. □

We will additionally consider functions $r(n,s)$ with the following property.

> *Property (P)*
> For each $n \in \mathbb{N}$ there exists $k = k(n) \in \mathbb{N}$ such that for all $s \in [0,1]$, $r(n,s) = r_{-2,k}(n,s)$. The function $n \mapsto k(n)$ satisfies $k(n) = o(\log\log(n))$. Furthermore, for each $k \in \mathbb{N}$ there exists $n_0 = n_0(k)$ such that for all $n \geq n_0$, if $r(n,s) = r_{-2,k'}(n,s)$ then $k' \geq k$.

Lemma 3.2. *If $r(n,s)$ has property (P), then $r(n,s)$ is a growth rate function.*

Proof. Since for each k, $r_{-2,k}$ is a growth rate function with $r_{-2,k}(n,1) = 2, r_{-2,k}(n,0) = n$, properties (1) and (2) hold for r. Furthermore, for each $s' > s \geq 0, n \in \mathbb{N}$ $\log\log r(n,s') = \beta \cdot \log\log r(n,s)$ for $\beta = \frac{1-s'}{1-s} < 1$, hence using $k(n) = o(\log\log(n))$, $\lim_{n\to\infty}(r(n,s') - r(n,s)) = -\infty$. □

4 Scaled Dimension of $\mathbf{ISO}_m^p(SAT)$

In this section we consider the class $ISO_m^p(SAT) = \{A \in NPC(\leq_m^p) | A \simeq_p SAT\}$. If we can show that there exists some positive integer k such that for each polynomial isomorphism f from some set $A \in NP$ to SAT, for each $s < 1$ for infinitely many n, a sufficient majority of all strings of length n are mapped to strings of length at most $k \cdot n$ under f, then we can use an $O(2^{cn})$-time bounded exhaustive search algorithm for SAT in order to obtain $\dim_p^{(r)}(\mathrm{ISO}_m^p(SAT)) = 0$. Hence we seek for some growth rate function r such that if such a condition is not satisfied by f, then f cannot be an isomorphism.

Theorem 4.1. *If $s \in (0,1)$ and r is a growth rate function such that for all $j \in \mathbb{N}$ there are infinitely many n such that $r(2^{n^j+1} - 1, s) < 2^n$, then $\dim_p^{(r)}(ISO_m^p(SAT)) \leq s$.*

Proof. Fix some $k \in \mathbb{N}$. Let $L \in NP$ and f be a a polynomial-time computable honest reduction from L to SAT. We will show that if for almost all n,

$$|\{x \in \{0,1\}^* |\, |x| \leq n,\ |f(x)| \leq k \cdot |x|\}| \ < \ r(2^{n+1} - 1, s) \quad (\star)$$

then f cannot be onto. There exists some $j \in \mathbb{N}$ such that $f(x)$ can be computed in time $|x|^j$ and $|x|^{1/j} \leq |f(x)| \leq |x|^j$ for all $x \in \{0,1\}^*$. Assume $(\star)$ holds for almost all n. Then for appropriate n large enough, the number of strings being mapped to strings of length n under f is upper-bounded by $r(2^{n^j+1} - 1, s) < 2^n$ which implies that f cannot be onto. Therefore, if f is a polynomial-time isomorphism from L to SAT, then for infinitely many n,

$$|\{x \in \{0,1\}^* |\, |x| \leq n,\ |f(x)| \leq k \cdot |x|\}| \ \geq \ r(2^{n+1} - 1, s)$$

and hence we can use the fact $SAT \in E$ in order to construct a polynomial-time computable martingale d that $r(n,s)$-succeeds on $\mathrm{ISO}_m^p(SAT)$. Let $\{f_j | j \in \mathbb{N}\}$ be an enumeration of all polynomial-time computable functions. Let $Z_j = \{f_j^{-1}(SAT)\}$ iff f_j is a polynomial-time computable, polynomial-time invertible bijection and $Z_j = \emptyset$ otherwise. For j such that $Z_j \neq \emptyset$, we will define a martingale d_j for Z_j as follows:

$$d_j(xb) = \begin{cases} 2 \cdot d_j(x) & \text{if } |f(z_{|x|})| \leq k \cdot |z_{|x|}|,\ b = SAT(f(z_{|x|})) \\ 0 & \text{if } |f(z_{|x|})| \leq k \cdot |z_{|x|}|,\ b = 1 - SAT(f(z_{|x|})) \\ d_j(x) & \text{otherwise} \end{cases}$$

Obviously the d_j are uniformly computable in time $O(2^{cn})$ for some $c > 0$. Furthermore, for each j there are infinitely many $n \in \mathbb{N}$ such that $d(f_j^{-1}(SAT)|n) \geq 2^{r(n,s)}$. This establishes $\dim_p^{(r)}(\mathrm{ISO}_m^p(SAT)) \leq s$. □

Corollary 4.1. *If r is a growth rate function that has property (P), then*

$$\dim_p^{(r)}(ISO_m^p(SAT)) = 0.$$

Proof. Let r have property (P). Then for given s and j, for n such that $k(n) = k$,

$$r(2^{n^j+1} - 1, s) = 2^{(\log(2^{n^j+1}-1))^{\frac{1-s}{k}}} \leq 2^{(\log(2^{n^{2j}}))^{\frac{1-s}{k}}} = 2^{n^{\frac{2j(1-s)}{k}}}$$

hence if we choose n large enough such that $k > 2j$, the condition of Theorem 4.1 is satisfied by r. □

5 On Disjoint Unions of NP Languages

Given two sets $A, B \subseteq \{0,1\}^*$, their disjoint union $A \oplus B$ is defined as

$$A \oplus B = \{a0 \mid a \in A\} \cup \{b1 \mid b \in B\}$$

Note that $\{L_1 \oplus L_2 | L_1, L_2 \in NP\} = NP$ if we consider NP as a subset of the power set of $\{0,1\}^*$. In this section we are concerned with subsets of $\{L_1 \oplus L_2 | L_1, L_2 \in NP\}$ and their scaled dimensions. First consider the set

$$\{L_1 \oplus L_2 \mid L_1, L_2 \in NP,\ L_1 \equiv_m^p L_2\}$$

(the set of all pairs of languages from NP that are polytime many-one equivalent).

Lemma 5.1. *Let $\mathcal{C}$ be a class of languages $L \subseteq \{0,1\}^*$ and $\mathcal{D}$ be a class of disjoint unions of languages that contains $\{L \oplus L | L \in \mathcal{C}\}$ Then for every growth rate function r such that for almost all n, $r(2n, s) - n \geq r(n, t)$, $\dim_p^{(r)}(\mathcal{D}) < s$ implies $\dim_p^{(r)}(\mathcal{C}) \leq t$.*

Proof. Assume d is a polytime computable martingale s-succeeding on $\mathcal{D}$, hence especially on the set of all $L \oplus L$, $L \in \mathcal{C}$. Now we define a martingale d' as follows: Let

$$\begin{aligned} d(x_1x_1 \ldots x_{n-1}x_{n-1}\,0) &= \alpha \cdot d(x_1x_1 \ldots x_{n-1}x_{n-1}), & \beta = 2 - \alpha \\ d(x_1x_1 \ldots x_{n-1}x_{n-1}\,00) &= \alpha_1 \cdot d(x_1x_1 \ldots x_{n-1}x_{n-1}\,0), & \beta_1 = 2 - \alpha_1 \\ d(x_1x_1 \ldots x_{n-1}x_{n-1}\,10) &= \alpha_2 \cdot d(x_1x_1 \ldots x_{n-1}x_{n-1}\,1), & \beta_2 = 2 - \alpha_2 \end{aligned}$$

Then we define

$$\begin{aligned} d'(x_1 \dots x_{n-1}0) &:= \tfrac{2\cdot\alpha\cdot\alpha_1}{\alpha\cdot\alpha_1+\beta\cdot\beta_2}\cdot d'(x_1\dots x_{n-1}) \\ d'(x_1 \dots x_{n-1}1) &:= \tfrac{2\cdot\beta\cdot\beta_2}{\alpha\cdot\alpha_1+\beta\cdot\beta_2}\cdot d'(x_1\dots x_{n-1}) \end{aligned}$$

By definition and the martingale property of d,

$$\frac{2}{\alpha\cdot\alpha_1+\beta\cdot\beta_2} \geq \frac{2}{4} = \frac{1}{2}$$

Hence if the capital of d on $L \oplus L$ is growing by factor $f \in \{\alpha\cdot\alpha_1, \beta\cdot\beta_2\}$, then the capital of d' on L is growing by at least $\frac{f}{2}$. Thus from $d(L\oplus L|1\dots 2N) \geq 2^{r(2N,s)}$ we obtain $d'(L \mid 1\dots N) \geq 2^{-N}\cdot 2^{r(2N,s)} = 2^{r(2N,s)-N}$ and hence the lemma follows. □

Corollary 5.1. *For each* $s \in (0,1)$*,* $\dim_p(NP) \geq 2s$ *implies* $\dim_p(\{L_1 \oplus L_2 \mid L_1, L_2 \in NP,\ L_1 \equiv^p_m L_2\}) \geq s$*.*

Proof. For $r_0(n,s) = (1-s)\cdot n$ we obtain $r_0(2N,s) - N = (1-2s)N$, hence $\dim_p(NP) \geq 2s$ implies $\dim_p(\{L_1 \oplus L_2 \mid L_1, L_2 \in NP,\ L_1 \equiv^p_m L_2\}) \geq s$. □

We will now consider disjoint unions of sets from NP that are in the same polynomial-time isomorphism degree.

Given $L \in NP$, we define the *even part* $L_e = \{x|x0 \in L\}$ and the *odd part* $L_o = \{x|x1 \in L$ of L. By definition, $L \setminus \{\Lambda\} = L_e \oplus L_o$. Now we apply two operations to L_e and L_o. The first is an operator considered already in [Hitchcock] and [Ambos-Spiess et al.] which makes them NP-complete, the second - which does not affect the first - makes them polynomially isomorphic. We are able to define these operations in such a way that they produce only "minor changes", hence we will be able to relate scaled dimensions of the set

$$\{L_1 \oplus L_2 \mid L_1, L_2 \in \deg^p_m(SAT),\ L_1 \simeq_p L_2\}$$

to the scaled dimensions of NP. We will now describe the two steps in detail.

First Step: Let F_k denote the map such that for every $n \in \mathbb{N}$, F_k maps all the strings of length n to the first 2^n strings of length n^k:

$$F_k\colon \{0,1\}^*,\ \ F_k(x) = x0^{|x|^k-|x|}$$

Given some $L = L_e \oplus L_o$, the operator $\mathcal{F}_k\colon \{0,1\}^\omega \to \{0,1\}^\omega$ applies the map F_k to L_e and L_o separately in order to encode a stretched version of SAT into them: $\mathcal{F}_k(L) = \mathcal{F}'_k(L_e) \oplus \mathcal{F}'_k(L_o)$ with

$$\mathcal{F}'_k(L_j) = (L_j \setminus F_k(\{0,1\}^*) \cup F_k(SAT) \ \ (j \in \{e,o\})$$

Note that for every $L \in NP$, $\mathcal{F}_k(L) \in \{L_1 \oplus L_1 | L_1, L_2 \in \deg^p_m(SAT)\}$.

Second Step: Given a disjoint union $L_1 \oplus L_2$ of two sets $L_1, L_2 \subseteq \{0,1\}^*$, the operator $\mathcal{G}_k$ encodes L_1 into L_2 and L_2 into L_1 by encoding the strings of length n into the second 2^n strings of length n^k. Note that this does not affect the previous application of operator

$\mathcal{F}_k$. More precisely: Let $g_k: \{0,1\}^* \to \{0,1\}^*$ be defined by $g_k(x) = x10^{|x|^k - |x| - 1}$ Let $\mathcal{G}_k(L_1 \oplus L_2) = G'_k(L_1, L_2) \oplus G'_k(L_2, L_1)$ with

$$G'_k(A, B) \;=\; (A \setminus g_k(\{0,1\}^*)) \cup g_k(B)$$

Now let

$$\mathcal{H}_k: \{L_1 \oplus L_2 | L_1, L_2 \in NP\} \to \{L_1 \oplus L_2 | L_1, L_2 \in \deg_m^p(SAT),\ L_1 \simeq_p L_2\}$$
$$\mathcal{H}_k(L_1 \oplus L_2) \;=\; \mathcal{G}_k(\mathcal{F}_k(L_1 \oplus L_2))$$

Analysis. Given some martingale d succeeding on

$$\{L_1 \oplus L_2 \mid L_1, L_2 \in \deg_m^p(SAT),\ L_1 \simeq_p L_2\},$$

we define the martingale d_k - informally - as follows: On the first $4 \cdot 2^n$ strings of length $n^k + 1$ it behaves neutrally (recall: $2 \cdot 2^n$ are needed for making L_e and L_o polynomial-time isomorphic, $2 \cdot 2^n$ are needed for making them NP-complete). On all the other positions, d_k behaves in the same way as d does.

Hence if $L' = \mathcal{H}_k(L)$, then $d_k(L|1 \ldots 2^{2n^k}) \;\geq\; \frac{d(L'|1 \ldots 2^{2n^k})}{2^{2 \cdot 2^n}}$. We conclude that if r is a growth rate function such that for almost all n, $r(2^{2n^k}, s) \;\geq\; 2 \cdot 2^n \,+\, r(2^{2n^k}, t)\;\; (s < t)$, then NP and $\{L_1 \oplus L_2 | L_1, L_2 \in \deg_m^p(SAT),\ L_1 \simeq_p L_2\}$ have the same r-scaled dimension. Especially we obtain the following result.

Theorem 5.1. *For each growth rate function* $r \in \{r_i \mid |i| \leq 2\} \cup \{r_{-2,k} | k \in \mathbb{N}\}$,

$$\dim_p^{(r)}(\{L_1 \oplus L_2 \mid L_1, L_2 \in deg_m^p(SAT),\ L_1 \simeq_p L_2\}) \;=\; \dim_p^{(r)}(NP)$$

On the other hand, using the methods from section 4 we obtain the following result (cf. Theorem 4.1).

Theorem 5.2. *Let r be a growth rate function. If r has property (P), then*

$$\dim_p^{(r)}(\{L_1 \oplus L_2 \mid L_1, L_2 \in deg_m^p(SAT),\ L_1 \simeq_p L_2\}) \;=\; 0.$$

References

[AAI+01] A. Agrawal, E. Allender, R. Impagliazzo, T. Pitassi, and S. Rudich. Reducing the complexity of reductions. *Computational Complexity*, 10(2):117–138, 2001.

[AER98] M. Agrawal, E.Allender, and S. Rudich. Reductions in circuit complexity: An isomorphism theorem and a gap theorem. *Journal of Computer and System Sciences*, 57(2):127–143, 1998.

[Agr02] M. Agrawal. Pseudorandom generators and structure of complete degrees. In *17th Annual IEEE Conference on Computational Complexity*, pages 139–145, 2002.

[ASMRS01] K. Ambos-Spies, W. Merkle, J. Reimann, and F. Stephan. Hausdorff dimension in exponential time. In *Proceedings of the 16th IEEE Conference on Computational Complexity*, pages 210–217, 2001.

[BH77] L. Berman and J. Hartmanis. On isomorphism and density of np and other complete sets. *SIAM Journal on Computing*, 6(2):305–322, 1977.

[HHP07] Ryan C. Harkins, J. M. Hitchcock, and A. Pavan. Strong Reductions and Isomorphism of Complete Sets. *Proceedings of the 27th International Conference on Foundations of Software Technology and Theoretical Computer Science. Lecture Notes in Computer Science* 4855:168–178, Springer-Verlag, 2007.

[Hit04] John M. Hitchcock. Small spans in scaled dimension. *SIAM Journal on Computing*, 34(1):170–194, 2004.

[HLM] John M. Hitchcock, Jack H. Lutz, and Elvira Mayordomo. Scaled dimension and nonuniform complexity. *Journal of Computer and System Sciences*, 69(2):97-122, 2004.

[HP07] John M. Hitchcock and A. Pavan. Comparing reductions to np-complete sets. *Information and Computation*, 205(5):694–706, 2007.

[HLVM] John M. Hitchcock, Mara Lopez-Valdes, and Elvira Mayordomo. Scaled dimension and the Kolmogorov complexity of turing hard sets. *Theory of Computing Systems*, 43(3-4):471-497, 2008.

[HPV] John M. Hitchcock, A. Pavan, , and N. V. Vinodchandran. Partial Bi-Immunity, Scaled Dimension, and NP-Completeness. *Theory of Computing Systems*, 42(2):131-142, 2008.

[JL95] D. W. Juedes and J. H. Lutz. The complexity and distribution of hard problems. *SIAM Journal on Computing*, 24(2):279–295, 1995.

[KMR89] Stuart A. Kurtz, Stephen R. Mahaney, and James S. Royer. The isomorphism conjecture fails relative to a random oracle (extended abstract). In *ACM Symposium on Theory of Computing*, pages 157–166, 1989.

[Lut00] Jack H. Lutz. Dimension in complexity classes. In *IEEE Conference on Computational Complexity*, pages 158–169, 2000.

[Lut] Jack H. Lutz. Effective fractal dimensions. *Mathematical Logic Quarterly*, 51:62-72, 2005.

[LM99] Jack H. Lutz and Elvira Mayordomo. Twelve problems in resource-bounded measure. *Computational Complexity Column (E. Allender, ed.), Bulletin of the European Association for Theoretical Computer Science*, 68:64–80, 1999.

[Rog95] John Rogers. The isomorphism conjecture holds and one-way functions exist relative to an oracle. In *Structure in Complexity Theory Conference*, pages 90–101, 1995.