Trading Tensors for Cloning: Constant Time Approximation Schemes for Metric MAX-CSP

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Abstract

We construct the first constant time value approximation schemes (*CTASs*) for Metric and Quasi-Metric MAX-rCSP problems for any $r \geq 2$ in a preprocessed metric model of computation, improving over the previous results of [FKKV05] proven for the general core-dense MAX-rCSP problems. They entail also the first *sublinear* approximation schemes for constructing approximate solutions of the above optimization problems.

1 Introduction

In [FKKV05] a general result was proved on existence of PTAS for *core dense* MAX-CSP problems. The result depends on a new method of approximating a tensor by the sum of small number of rank-1 tensors similar to the traditional Singular Value Decomposition. In this paper we are going to construct more efficient (in fact, constant time) approximation schemes for the special case of metric and quasimetric instances of the MAX-CSP problems.

Assume that r, n are integers where $r \geq 2$ is fixed. Let $V = \{v_1, v_2, \ldots, v_n\}$ be a set of boolean variables.

An instance of Metric MAX-rCSP, a natural generalization of Metric MAX-2CSP, is defined by a pair (F, d) where $F = \{f_1, f_2, \ldots, f_m\}$ is a set of boolean functions depending each on exactly r variables in V and d is a metric defined on

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V. We let $f^{(j)}_{i_1,i_2,\ldots,i_r}$ for $j \in I_{i_1,i_2,\ldots,i_r}$, denote the functions involving the variables $i_1, i_2, \ldots, i_r$. We assume that F comprises at least one function for each r-set $\{i_1, i_2, \ldots, i_r\} \in \binom{V}{r}$. For each r-set $\{i_1, i_2, \ldots, i_r\}$ the weight of each of the functions $f^{(j)}_{i_1,i_2,\ldots,i_r}$ is defined to be the sum of the $\binom{r}{2}$ pairwise distances:

$$W_{i_1,i_2,\ldots,i_r} = \sum_{1\le k<j\le r} d(v_{i_k}, v_{i_j}) .$$

We are to find a boolean assignment A to the variables $v_1, v_2, \ldots, v_n$ which maximizes the sum

$$\sum_{i_1,i_2,\ldots,i_r} \left(W_{i_1,i_2,\ldots,i_r} \sum_{j\in I_{i_1,i_2,\ldots,i_r}} f^{(j)}_{i_1,i_2,\ldots,i_r} \right)$$

of the weights of functions which are satisfied by A.

We assume that d is scaled so that the average of the $n(n-1)/2$ distances between points of V is 1. We define for each i

$$w_i = \sum_j d(v_i, v_j).$$

2 Results and Technique Overview

Metric MAX-rCSPs are core-dense in the sense of [FKKV05], and thus they have PTASs running in time $n^{O(1/\epsilon^2)}$ for relative accuracy ϵ (see [FKKV05]). We give here, by applying ideas of *cloning* (cf. [FK01]), a solution value PTAS running in constant time $2^{O^{\sim}(1/\epsilon^2)}$ in a preprocessed metric model of computation. This yields a sublinear solution-constructing PTAS working in time $2^{O^{\sim}(1/\epsilon^2)} \cdot n^{r-1}$ for Metric MAX-rCSP problems for arbitrary r. This improves also an original MAX-CUT approximation scheme of [FK01].

We use in our construction a cloning method introduced in [FK01], and a fast approximate computation of the metric weights due to Indyk [I99b]. Then, we use the main result of [AFKK02] which states, roughly speaking, that the value of a MAX-rCSP is w.h.p. approximately equal to the value of the problem induced on a random subset of the variables of size $\Omega(\log(1/\epsilon)/\epsilon^4)$ times a scaling factor.

Our results easily extend to the case of general quasimetrics (see also [FKKV05], [MS79]) which include important for various applications powers of arbitrary metrics.

3 Model of Computation

Given a metric space (V, d), notice that the size of the input describing (V, d) is $\Theta(n^2)$. We consider two models of computation in the metric (or quasimetric) spaces. First, the *Preprocessed Model* where the sums of weights w_i are being precomputed and given by an oracle. (Recall that w_i is the sum of the distances to v_i.) In the second model, we *compute* approximate values of the w_i following Indyk [I99a]. This can be done approximately in time $O(n\mathrm{poly}\,(\log n))$.

4 Main Results

We formulate now our main results. We note that sizes of inputs for MAX-rCSP problems are $\Theta(n^r)$, and thus running times in $o(n^r)$ are *sublinear* in the input sizes.

Theorem 1. *There exist constant time $2^{O^{\sim}(1/\epsilon^2)}$ approximation schemes (CTASs) in the preprocessed model of computation for estimating the optimum value of metric and quasimetric Max-rCSPs for any $r \geq 2$.*

By approximate implementation of the preprocessed metric model we obtain

Theorem 2. *There exists sublinear time value approximation schemes working in time $2^{O^{\sim}(1/\epsilon^2)} + O\big(n\mathrm{poly}\left(\frac{1}{\epsilon}\log n\right)\big)$ for metric and quasimetric Max-rCSPs for any $r \geq 2$.*

Using a method of Section 9, we are able to formulate a result on constructing approximate solution-assignments for Max-rCSP problems.

Theorem 3. *There exists sublinear time approximation schemes working in time $2^{O^{\sim}(1/\epsilon^2)}n^{r-1} + O\big(n\mathrm{poly}\left(\frac{1}{\epsilon}\log n\right)\big)$ for metric and quasimetric Max-rCSPs for any $r \geq 2$.*

The proofs of Theorems 1-3 are given in the following sections of the paper.

5 Cloning

The main idea of our CTASs for Metric MAX-rCSP, is that of *cloning* similar to [FK01], i.e. constructing a new MAX-rCSP problem $(\tilde{F}, \tilde{W})$ by replacing each variable v_i by a certain number m_i, say, of copies $v_{i,1}, v_{i,2}, ..., v_{i,m_i}$, called *clones*. For each $\{i, j, ..., \ell\} \in \binom{V}{r}$, $\tilde{F}$ will comprise $m_i m_j ... m_\ell$ functions identical to $f_{i,j,...,\ell}$ and each acting on a particular r-tuple of clones of the form $v_{i,s}, v_{j,t}, ..., v_{\ell,u}$. We take in fact

$$m_i = \lceil w_i \rceil.$$

Let us denote by $\tilde{V}$ the new set of variables. Now we assign to all the r-tuples of the form $v_{i,s}, v_{j,t}, ..., v_{\ell,u}$ for fixed $i, j, ..., \ell$ the same weight denoted by $\tilde{W}_{i,j,...,\ell}$:

$$\tilde{W}_{i,j,...,\ell} = \frac{W_{i,j,...,\ell}}{m_i m_j ... m_\ell}$$

We end up in this way, as we prove in the next section, with a *dense weighted* instance in the sense of [FK00] for which we can use known approximation algorithms. Note that here as in [FK01] *cloning* is just a convenient disguised form of a special *weighted* sampling.

6 Cloned Instances are Weighted Dense

In this section, we prove that the instances $(\tilde{F}, \tilde{W})$ are *dense* in the sense that the maximum weight of a constraint does not exceed the average of the weights by more than a constant factor. We will use as in [FK01] the inequalities

$$w_u \geq \frac{n}{2}. \tag{1}$$

$$d(u,v) \leq \frac{w_u + w_v}{n}. \tag{2}$$

Since each pair of vertices $\{v_i, v_j\}$ belongs to precisely $r!\binom{n-2}{r-2}$ $r - sets$, the sum S, say, of the weights in the original instance:

$$\begin{aligned} S &= r!\binom{n-2}{r-2} \sum_{1 \leq j < k \leq n} d(v_i, v_j)\,, \\ S &\sim \frac{r(r-1)n^r}{2} \end{aligned}$$

the last because the sum of the distances is $\binom{n}{2}$. The sum of the weights in the cloned instance, say S' would be the same as S if we had $m_i = w_i$. From our choice $m_i = \lceil w_i \rceil$, it follows that we have $S' = S(1 + O(1/n))$ and

$$S' \sim \frac{r(r-1)n^r}{2}\,. \tag{3}$$

Now the number of functions in $\tilde{F}$ is

$$\begin{aligned} |\tilde{F}| &= \sum_{(i_1,i_2,\ldots,i_r)\in V^r} m_{i_1} m_{i_2} \ldots m_{i_r} \\ &\leq 2 \sum_{(i_1,i_2,\ldots,i_r)\in V^r} w_{i_1} w_{i_2} \ldots w_{i_r} \\ &\leq 2 \left(\sum_{u \in V} W_u \right)^r \end{aligned}$$

$$|\tilde{F}| \leq 2n^{2r} \tag{4}$$

Upon dividing, we get that the mean weight in $\tilde{F}$ is bounded below by

$$\frac{r(r-1)}{4n^r}$$

We denote by c the maximum weight. c is clearly bounded above by the maximum over all the choices of $i_1, i_2, ..., i_r$ of the ratio

$$\frac{\sum_{j,k \in \{i_1, i_2, ..., i_r\}} d(v_{i_j}, v_{i_k})}{w_{i_1} w_{i_2} ... w_{i_r}}$$

By (2) we get that

$$c \leq \frac{(r-1) \sum_{j=1}^{r} w_{i_j}}{n\, w_{i_1} w_{i_2} ... w_{i_r}}$$

and, since w_i is at least $n/2$, we get

$$c \leq \frac{r(r-1)2^{r-1}}{n^r} \tag{5}$$

Using the previous bound for the mean weight, we get that the ratio of this maximum to the average does not exceed 2^{r+1}. (Our computations give actually the bound $2^r(1 + o(1))$ as n tends to infinity.)

7 Cloned Metric MAX-rCSPs Are Optimized by Pure Assignments

Call an assignment to $\tilde{V}$ *pure*, if for each $1 \leq i \leq n$ all the clones $v_{i,1}, v_{i,2}, \ldots, v_{i,m_i}$ of v_i are assigned to the same truth value. A pure assignment defines in the obvious way a solution to the original problem (F, W) with the same value as the solution it defines on $\left(\tilde{F}, \tilde{W}\right)$.

For an assignment A to $\tilde{V}$, we denote by $val(A)$ the corresponding value of the objective function in the instance $\left(\tilde{F}, \tilde{W}\right)$. The following claim implies immediately the assertion of the title of this section.

Claim: *Let $A = \tilde{V} \to \{0, 1\}$ be an assignment to $\tilde{V}$. Assume that A is not pure for the variable v_1. Let $A^{(o)}$ (resp. $A^{(1)}$) be the assignment obtained from A by assigning all the clones of v_1 to 0 (resp. to 1) and keeping A unmodified elsewhere. Then one of $val(A^{(o)})$ and $val(A^{(1)})$ is at least $val(A)$.*

For the proof, recall that m_1 denotes the number of clones of v_1. For each $j \in \{1, 2, \ldots, k_1\}$ the set of clauses in the disjunctive normal form of F containing $v_{1,j}$ is of the form

$$\{v_{1,j} \wedge C : C \in \mathcal{C}_1\}$$

say, where $\mathcal{C}_1$ is a certain set of $(r-1)$- conjunctions which does not depend on j. Similarly, the set of clauses in the disjunctive normal form of F containing $\bar{v}_{1,j}$ is of the form

$$\{\bar{v}_{1,j} \wedge D : D \in \mathcal{D}_1\}$$

say, where $\mathcal{D}_1$ is a certain set of $(r-1)$- conjunctions which does not depend on j. (This is because $\left(\tilde{F}, \tilde{W}\right)$ is invariant when we interchange $v_{1,j}$ and $v_{1,k}$, $k \neq j$)

Write:

- c_1 for the weighted number of conjunctions $C \in \mathcal{C}_1$ true under A where each C has the weight of $(v_{1,1} \wedge C)$

- d_1 for the weighted number of conjunctions $D \in \mathcal{D}_1$ true under A where each D has the weight of $(\bar{v}_{1,1} \wedge D)$ - $n_1^{(0)}$ for the number of $v_{1,j}$ assigned to 0 by A

- $n_1^{(1)}$ for the number of $v_{1,j}$ assigned to 1 by A

- $A^{(res)}$ for the restriction of A to the set $\tilde{V} \backslash \{v_{1,1}, v_{1,2}, \ldots, v_{1,k_1}\}$

We have then that:

$$val(A^{(0)}) - val(A^{(res)}) = m_1 c_1 \tag{6}$$

$$val(A^{(1)}) - val(A^{(res)}) = m_1 d_1 \tag{7}$$

$$val(A) - val(A^{(res)}) = n_1^{(0)} c_1 + n_1^{(1)} d_1 \tag{8}$$

Since $m_1 = n_1^{(0)} + n_1^{(1)}$ it is clear than one of 6 and 7 is at least as big as 8.

8 The PTASs

We apply in this section an extension of the results of [AFKK02] (see for the background results also [AKK95], [F96], and [FK97]).

Let $F = \{f_1, f_2, \ldots, f_m\}$ be a set of m distinct boolean functions of n variables $v_1, v_2, \ldots, v_n$ each involving r of the variables and let $a_1, a_2, \ldots, a_m$ be non-negative weights bounded by b, say, where b does not depend on n. We let $Max(F)$ denote the maximum weighted number of functions which can be satisfied by a truth assignment to the variables, where f_i has the weight a_i. For a subset Q of the variables we let F^Q denote the subset of F which are functions of only variables in Q.

Theorem 1 of [AFKK02] has been stated without weights. We generalize it to the above case of non-negative weights $a_1, a_2, \ldots, a_m$. The proof carries through directly to that weighted situation. (We did not attempt the strongest possible form of the theorem here.)

Theorem 4. *Let r,n, be positive integers, with r fixed. Suppose ϵ is a positive real. There exists a positive integer $q \in O(\log(1/\epsilon)/\epsilon^4)$ such that for any F as above, if Q is a random subset of $\{v_1, v_2, \ldots, v_n\}$ of cardinality q, then with probability at least 9/10, we have*

$$\left| \frac{n^r}{q^r} Max(F^Q) - Max(F) \right| \leq \epsilon n^r .$$

By applying Theorem 4 to our weighted instance $(\tilde{F}, \tilde{W})$ and computing $\mathrm{Max}(F^Q)$ by exhaustive search we get an approximation to the optimum value within ϵn^r. Now, by the preceding section we know that there is a pure solution at least as good as the approximation we have. (Note that we do not compute such an assignment.) This pure solution induces in the obvious way an approximation to the original instance (F, W) with the same relative error. Adding the easy observation that the optimum of $(\tilde{F}, \tilde{W})$ rescaled to average weight 1 gives constant time approximation scheme working in time $2^{O^{\sim}(1/\epsilon^2)}$ (cf. [AFKK02]) provided an oracle gives us approximate values of the w_i.

Without oracle the overall time is dominated by the time needed for the approximate computation of the w_i. The later is in $O(npoly(\log n))$ by a result of Indyk [I99a].

9 Extracting Assignments from Solution Values: Proof of Theorem 3

Recall that Theorem 3 asserts the following.

There exists sublinear time approximation schemes working in time $2^{O^{\sim}(1/\epsilon^2)} n^{r-1} + O\big(n\text{poly}\left(\frac{1}{\epsilon}\log n\right)\big)$ for metric and quasimetric Max-rCSPs for any $r \geq 2$.

Proof. We assume the preprocessed model of computation (see Section 3). By working on the space of clones (see Section 5) we can assume that the instance is dense. Now we claim the following:

Proposition. *Assume that we have an instance of MAX-rCSP defined by a collection of functions F. and assume we pick a random sample S of the r-sets of variables by choosing randomly each r-set with probability p, and let* $m = p\binom{n}{r}$. *Let G be the set of functions in F corresponding to these r-sets. Let* $\text{val}(F, A)$ *resp.* $\text{val}(G, A)$ *be the number of functions in F, resp. in G, true under the assignment A. If* $m = n^{r-1} f(n)$ *with* $f(n) = \omega(1)$*, then we have w.h.p.*

$$\max_A |\text{val}(F, A) - \frac{\binom{n}{r}}{m} \text{val}(G, A)| \leq \epsilon n^r$$

where the max is taken over the set of all assignments.

Proof. Fix an assignment A and let $Sat(A, F)$, resp. $Sat(A, G)$, denote the set of functions in f resp. G satisfied by A. Let $m = |Sat(A, F)|$. We have that

$$|Sat(A, G)| = \sum_{Y \in \binom{V}{r}} n_Y B_Y(1, p)$$

where n_Y is the number of functions of the r-tuple Y satisfied by A and the $B_Y(1, p)$ are Bernouilli variables each with parameter p Therefore, using the bound $n_Y \leq 2^{2^r}$ by Hoeffding we have that

$$\begin{aligned}\Pr(||Sat(A,G)| - mp| \geq \epsilon n^r) &\leq 2\exp\left(-\frac{2^{-2^r+1}\epsilon^2 n^{2r}}{mp}\right) \\ &\leq 2\exp\left(-\frac{2^{-2^r+1}\epsilon^2 n}{f}\right).\end{aligned}$$

Using the union bound we find that, for any fixed ϵ if $f = C(r)/\epsilon^2$, where $C(r)$ depends only on r, the event $||Sat(A,G)| - mp| \geq \epsilon n^r$ is true simultaneously for all assignments A with probability at least 3/4.

□

Theorem 3 follows almost immediately from the above proposition. We sample the cloned instance which can be done in the required time. Then we compute an assignment A for which the number of constraints in the set of G corresponding to the sample is approximately maximized. By the above proposition, A is also, with high probability, approximately maximizing for F. For let B be an optimal assignment for F and A an optimal assignment for G But we have that

$$\mathrm{val}(F,A) \geq \frac{\binom{n}{r}}{m}\mathrm{val}(G,A) - \epsilon(n^r)$$

and

$$\mathrm{val}(F,B) \leq \frac{\binom{n}{r}}{m}\mathrm{val}(G,B) + \epsilon(n^r)$$

With the previous inequality, this gives

$$\mathrm{val}(F,B) - \mathrm{val}(F,A) \leq 2\epsilon(n^r)$$

which shows that A is approximately optimal for F.

□

10 Some New Constructability Consequences

The results of Section 9 entail also the following improvements of hitherto known results for dense unweighted instances of Max-rCSP.

Corollary. *There exists sublinear time approximation schemes for constructing an almost optimal assignment for dense Max-rCSP problems working in time* $2^{\tilde{O}(1/\epsilon^2)}n^{r-1}$ *for any* $r \geq 2$.

We notice also that our results improve over the best known algorithms for Metric MAX-CUT (see [FK01] and [I99b]) and give for the first time sublinear approximation schemes for that problem.

Finally, our results can be extended to obtain sublinear approximation algorithms for constructing approximate solution-assignments for Metric Max- and Min-Bisection problems (see also [FKK04]).

An interesting open question remains about the existence of *sublinear* approximation schemes for the metric k-Clustering problems for arbitrary fixed k (see [FKKR03]).

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