Approximation of Global MAX-CSP Problems

W. Fernandez de la Vega * Ravi Kannan †

Marek Karpinski ‡

Abstract

We study the problem of absolute approximability of MAX-CSP problems with the global constraints. We prove existence of an efficient sampling method for the MAX-CSP class of problems with linear global constraints and bounded feasibility gap. It gives for the first time a polynomial in ϵ^{-1} sample complexity bound for that class of problems. The method yields also the best up to date sample complexity bound for the balanced MAX-CSP problems such as the graph and hypergraph BISECTION problems.

1 Introduction

We extend the results of [AFKK02] (see for the background results also [AKK95], [F96], [FK96], [FK99], [K01], [AN04], and [FK06]) to a class of global MAX-rCSP problems proving that they have also a polynomial in ϵ^{-1} sample complexity. The input to a MAX-rCSP problem (for r fixed) consists of a set F of m distinct Boolean functions $f_1, f_2, \ldots f_m$ of n Boolean variables $x_1, x_2, \ldots x_n$, where each f_i is a function of only r of the n variables. The output Max(F) is the maximum number of functions which can be simultaneously set to 1 by a truth assignment

*LRI, Université Paris-Sud, Orsay, Paris. Supported in part by PROCOPE Project and by IST grant 1496 (RAND-APX). Part of this work was done while visiting Dept. of Computer Science, Yale University. Email: lalo@lri.fr

†Dept. of Computer Science, Yale University. Supported in part by NSF Grant CCR-9820850. Email: kannan@cs.yale.edu

‡Dept. of Computer Science, University of Bonn. Supported in part by DFG grants, Max-Planck Research Prize, and by IST grant 14036 (RAND-APX). Part of this work was done while visiting Dept. of Computer Science, Yale University. Email: marek@cs.uni-bonn.de

to the variables. This paper addresses the situation where in addition to the above we have a set of linear global constraints:

$$Rx \leq p \tag{1}$$

where R is a $k \times n$ matrix with $||R||_\infty \leq 1$, and we want to compute approximately within additive error ϵn^r the maximum number of functions in F which can be simultaneously set to 1 by a truth assignment to the variables satisfying these constraints. The word global refers to constraints which may involve all the n variables instead of just r. We emphasize that *all* of these global constraints ought to be satisfied. If this is not possible, then we say as usual that the optimization problem is unfeasible. (In contradistinction, we ask that a maximum number of the other constraints be satisfied, not necessarily all of them.)

For a subset Q of the variables, we let F^Q denote the subset of those functions in F which depend only on the variables in Q (and their negations). We let x^Q denote an assignment to the variables in Q. For a $k \times n$ matrix R and a $k-$vector p we define $gap(R, p)$ by

$$gap(R, p) = \max\{\alpha : R^T x \leq p - \alpha n \mathbf{1} \text{ is feasible}\}$$

with $x \in [0, 1]^n$. ($\mathbf{1}$ denotes the all ones vector.) We will prove the following theorem.

Theorem 1. (Main Theorem) *Let r, n be positive integers, with r fixed. Suppose k is a fixed positive integer and δ is a fixed positive real. Then for any positive ϵ, there exists a positive integer $q \in O(\log(1/\epsilon)/\epsilon^4)$ such that for any F (as above), any set of k constraints $Rx \leq p$ with $gap(R, p) \geq \delta$ if Q is a random subset of $\{x_1, x_2, \ldots x_n\}$ of cardinality q, then with probability at least 9/10, we have,*

$$\left| \frac{n^r}{q^r} Max(F^Q) - Max(F) \right| \leq \epsilon n^r.$$

where $Max(F)$ is the maximum number of functions in F which can be simultaneously set to 1 by a truth assignment x to the variables satisfying the constraints $Rx \leq p$ and $Max(F^Q)$ is the maximum number of functions in F^Q which can be simultaneously set to 1 by a truth assignment to the variables Q satisfying the constraints

$$\frac{1}{q} R^Q x^Q \leq \frac{1}{n} p \tag{2}$$

Thus, using the terminology of [AFKK02] it is apt to say that MAX-rCSP with global constraints having lower bounded gap has a sample complexity in $O(\log(1/\epsilon)/\epsilon^4)$ which is the same upper bound as that which is known for the unconstrained case. We call the property that $gap(R, p) \geq \delta$ for some positive constant δ, a bounded feasibility *gap property* of that problem.

The plan of the rest of the paper as follows. In section 2 we prove an easy but essential proposition which says roughly that, under a certain mild slackness condition of the set of constraints, the value of the objective function does not vary too much under a small perturbation of the constraints. In section 3 we collect several theorems and results from [AFKK02] for further use. The proof is concluded, using arguments peculiar to the present case with global constrains in section 4.

2 Constraints Slackness

Assume we have the constraints $Rx \leq p$ where k is fixed and R is a $k \times n$ matrix. The next proposition says roughly that, under a certain *slackness* condition of the set of constraints, the value of the objective function does not vary too much under a small perturbation of the constraints.

Lemma 2. *Assume that gap*$(R, p) = \delta > 0$ *and that* x *satisfies*

$$\begin{aligned} Rx &\leq p + \eta n \mathbf{1} \\ x &\in [0,1]^n \end{aligned}$$

for some sufficiently small $\eta > 0$. *Then, there exist constants* C_1 *and* C_2 *(depending on* r, δ *but not on* η *or* n*) such that there is an* y *with* $y \in \{0,1\}^n$, $Ry \leq p$ *and*
(i) $|x - y| \leq C_1 \eta n$ *and*
(ii) the variation of the objective function when x *is changed into* y *does not exceed* $C_2 \eta n^r$.

Proof We find first an y in $[0,1]^n$. Randomized rounding gives then easily an integral y fulfilling the conditions of the theorem. (ii) follows easily from (i). To prove (i), let $z \in [0,1]^n$ satisfy $Rz \leq p - \delta n \mathbf{1}$ (the existence of such a z is guaranteed by the gap condition), i.e.,

$$R_i z = p_i - \delta_i \,, \quad 1 \leq i \leq k.$$

with $\delta_i \geq \delta$. Setting $y = \gamma z + (1 - \gamma)x$ we get for each fixed i,

$$\begin{aligned} R_i y &= (p_i + \eta_i)(1 - \gamma) + \gamma(p_i - \delta_i) \\ &\leq p_i + \eta_i - \gamma(\eta_i + \delta_i) \\ &\leq p_i \end{aligned}$$

for $\gamma \geq \eta_i / (\eta_i + \delta_i) \leq \eta / \delta$. Thus (ii) holds with $C_1 = 1/\delta$.

□

3 Preliminaries

We let $V = \{1, 2, \ldots, n\}$ and $r \geq 2$ be a fixed integer. A is an array on V^r. We mimic the proof in [AFKK02] for the case of no global constraints. This proof begins by the following cut decomposition theorem.

Theorem 3. *It is possible to find, in time* $2^{O(1/\epsilon^2)}O(N)$ *and with probability at least, say,* $9/10$*, a set of at most* $4/\epsilon^2$ *cut arrays whose sum, denoted* D*, satisfies the following inequalities :*

$$||A-D||_C \leq \epsilon\sqrt{N}||A||_F \tag{3}$$

$$||A-D||_F \leq ||A||_F \tag{4}$$

The sum of the absolute values of the coefficients of the cut arrays $\leq \dfrac{2||A||_F}{\epsilon\sqrt{N}}$. (5)

Proof See [AFKK02].

□

The following theorem is the crux of the proof in [AFKK02].

Theorem 4. *Suppose* G *is an* r*-dimensional array on* V^r *satisfying*

$$||G||_C \leq \epsilon n^r \quad ||G||_\infty \leq \frac{1}{\epsilon}2^{2^{r+1}} \qquad ||G||_F \leq 2^{2^r}n^{r/2}. \tag{6}$$

Let $\delta, \epsilon > 0$*. Assume* $n = |V| \geq \frac{10^8 r^{20}}{\delta^7\epsilon^8}e^{10/\epsilon^2}$*. Let* J *be a random subset of* V *of cardinality* q*, where,*

$$q \geq 10^6 r^{12}\frac{1}{\delta^5\epsilon^4}\log\left(\frac{4}{\epsilon^2}\right).$$

Let H *be the* r*-dimensional array obtained by restricting* G *to* J^r*. Then, we have with probability at least* $1-\delta$ *:*

$$||H||_C \leq 2^{2^{r+1}+9}\frac{\epsilon}{\sqrt{\delta}}q^r.$$

Proof See [AFKK02].

□

It is easy to prove (see [AFKK02]) that it suffices to maximize within ϵn^r the polynomial P over $\{0,1\}^n$:

$$P(x) = \sum_{z\in\{0,1\}^r}\sum_{i_1,i_2,\ldots i_r} A^{(z)}(i_1,i_2,\ldots i_r)\prod_{z_j=1}x_{i_j}\prod_{z_j=0}(1-x_{i_j}).$$

where for each $z \in \{0,1\}^r$, $A^{(z)}$ is an array on V^r, $A^{(z)}(i_1,i_2,...,i_r)$ is the number of functions in F which are made true by the assignment $x_{i_1} = z_1, \ldots x_{i_r} = z_r$. Trivially, get the analogous statement in the case of global constraints just by specifying that the max is then taken within the assignments satisfying the constraints. Now we use Theorem 3 to find a cut decomposition $B^{(z)}$ of each of the arrays $A^{(z)}$ where each $B^{(z)}$ is the sum of at most $4/\epsilon^2$ cut arrays and we have

$$||A^{(z)} - B^{(z)}||_C \leq \epsilon n^{r/2}||A^{(z)}||_F \leq \delta m/4. \qquad \forall z.$$

From this it is easily deduced (see again [AFKK02]) that it suffices to maximize the function $g(x)$ below to additive error ϵn^r :

$$g(x) = \sum_{z \in \{0,1\}^r} \sum_{i_1, i_2, \dots i_r} B^{(z)}(i_1, i_2, \dots i_r) \prod_{z_j=1} x_{i_j} \prod_{z_j=0} (1 - x_{i_j}). \tag{7}$$

This again carries over mutatis mutandis to the constrained case.

We will need in the sequel the following theorem ([AFKK02]) which asserts that for a Linear Program on n variables, each constrained to be between 0 and 1, we can make some assertion about the optimal value based on the optimal value of a small sub-program obtained by picking at random a small number of variables.

Theorem 5. *Suppose*

$$\alpha > Max \sum_{j=1}^{n} c_j x_j$$

$$\sum_{j=1}^{n} U_j x_j \leq v \qquad ; \qquad 0 \leq x_j \leq 1,$$

as before and, in addition,

$$\sum_{j=1}^{m} c_j^2 \leq \alpha_2 \qquad ||c||_\infty \leq M_2.$$

Suppose q *is a positive integer and* Q *is a random subset* $\{1, 2, \dots n\}$ *of cardinality* q*. Then, for any positive real number*

$$\gamma \in \left[\frac{4\alpha_2}{nM_2^2} \ , \ 100 \right],$$

we have that with probability at least $1 - 4e^{-\gamma q/4}$ *:*

$$\frac{q}{n}\alpha + 2\gamma q M_2 > Max \sum_{j \in Q} c_j x_j$$

$$\sum_{j \in Q} U_j x_j \leq \frac{q}{n} v - 2\sqrt{\gamma} q ||U||_\infty \qquad ; \qquad 0 \leq x_j \leq 1, j \in Q.$$

Proof See [AFKK02].

□

For convenience, we change the definition of $g(x)$ slightly here and normalize by dividing by n^r.

$$g(x) = \frac{1}{n^r} \sum_{z \in \{0,1\}^r} \sum_{i_1, i_2, \dots i_r} B^{(z)}(i_1, i_2, \dots i_r) \prod_{j: z_j=1} x_{i_j} \prod_{j: z_j=0} (1 - x_{i_j}).$$

Define

$$\tilde{g}(x) = \frac{1}{q^r} \sum_{z \in \{0,1\}^r} \sum_{i_1, i_2, \dots i_r \in Q} B^{(z)}(i_1, i_2, \dots i_r) \prod_{j: z_j=1} x_{i_j} \prod_{j: z_j=0} (1 - x_{i_j}). \tag{8}$$

$$E_2 \Longrightarrow \max_{\{x_j \in \{0,1\}, j \in Q\}} |g(x) - \tilde{g}(x)| \in O(\epsilon). \tag{9}$$

The next Lemma provides piecewise linear approximations to $g(x)$ and $\tilde{g}(x)$. We need some preparation before stating this Lemma. Suppose that the sets involved in defining all the cut arrays in the approximations of all $B^{(z)}$ are $S_1, S_2, \ldots S_s$. Denote by $h(x)$ the $s-$ vector $(\frac{1}{n}x(S_1), \frac{1}{n}x(S_2), \ldots \frac{1}{n}x(S_s))$ (for an $n-$ vector x) and similarly by $\tilde{h}(x)$ the $s-$ vector $(\frac{1}{q}x(S_1 \cap Q), \frac{1}{q}x(S_2 \cap Q), \ldots \frac{1}{q}x(S_s \cap Q))$ (for a $q-$ vector x with components for each $j \in Q$) and note that $h(x)$ (resp. $\tilde{h}(x)$) determine $g(x)$ (resp. $\tilde{g}(x)$) We approximate $g(x)$ by a piece-wise linear function, where, each piece will comprise of all the x 's for which the $h(x)$ are close. More precisely, we will use a parameter η - which will be $\Theta(\frac{\epsilon}{k})$ where k is the number of global constraints. Let $\mathcal{A}$ be the set of integer multiples of η in the range $(0, 1)$. For each $b \in \mathcal{A}^s$, define

$$I(b, \eta) = \{x : |h(x) - b|_\infty \leq 2\eta; \frac{1}{n}Rx \leq \frac{1}{n}p + \eta\mathbf{1}\}$$

$$\tilde{I}(b, \eta) = \{x^Q : |\tilde{h}(x^Q) - b|_\infty \leq \eta; \frac{1}{q}R^Q x^Q \leq \frac{1}{q}p + \eta\mathbf{1}\}$$

We will need also need the following simple fact which follows from the standard Martingale inequality:

Claim 1

$$E_1 : \left|\frac{1}{n}|S_t| - \frac{1}{q}|S_t \cap Q|\right| \leq \epsilon^2 \textit{ for} \quad t = 1, 2, \ldots s \quad \mathbf{Pr}(E_1) \geq 1 - 4se^{-\epsilon^4 q/4}.$$

We will denote by $\tilde{A}^{(z)}$ the sub-array of $A^{(z)}$ on Q^r. Similarly for $B^{(z)}$. From Theorem 4, (with $q = c\frac{1}{\epsilon^4}\log(1/\epsilon)$ for a high enough constant c), we see that the following event has the claimed probability :

$$E_2 : ||\tilde{A}^{(z)} - \tilde{B}^{(z)}||_C \in O(\epsilon q^r) \qquad \text{satisfies} \qquad \mathbf{Pr}(E_2) \geq \frac{99}{100}. \tag{10}$$

Lemma 6. *For a suitable choice of $\eta \in \Theta(\epsilon)$, for each fixed $b \in \mathcal{A}^s$, there exist two linear functions $l(x) = l_0 + \sum_{j=1}^n l_j x_j$ and $\tilde{l}(x) = \tilde{l}_0 + \sum_{j \in Q} \tilde{l}_j x_j$ such that*

$$|g(x) - l(x)| \in O(\epsilon)\ \forall x \in I(b, \eta) \qquad |\tilde{g}(x) - \tilde{l}(x)| \in O(\epsilon),\ \forall x \in \tilde{I}(b, \eta).$$

$$E_1 \text{ and } E_2 \Longrightarrow \left|\tilde{l}_0 + \sum_{j \in Q} \tilde{l}_j x_j - l_0 - \frac{n}{q}\sum_{j \in Q} l_j x_j\right| \in O(\epsilon) \forall x \in \tilde{I}(b, \eta).$$

Also, $|l_j| \in O(1/n\epsilon) \quad \forall j$ *and* $\sum_{j=1}^n l_j^2 \in O(1/n)$.

Proof See [AFKK02].

□

4 Proof of Main Theorem

Recall that $\tilde{A}^{(z)}$ denotes the sub-array of $A^{(z)}$ on Q^r and similarly for $B^{(z)}$. We have that

$$E_2 \Longrightarrow \max_{\{x_j\in\{0,1\},j\in Q\}} |g(x)-\tilde{g}(x)| \in O(\epsilon). \tag{11}$$

Thus, to prove Theorem 1, it suffices to show that

$$\left|\text{Max}_{x_j\in\{0,1\},j\in V} \quad g(x) - \text{Max}_{x_j\in\{0.1\},j\in Q} \quad \tilde{g}(x)\right| \in O(\epsilon). \tag{12}$$

Let Max(F) denote the max of F subject to $Rx \le p$ and let Max$(F) = \alpha n^r$. Clearly, for each b, the maximum value of the following Integer Program is at most $\alpha + O(\epsilon)$:

$$\begin{aligned} &\text{Max } l_0 + l_1x_1 + l_2x_2 + \dots l_nx_n \\ b_t - 2\eta \le \frac{1}{n}x(S_t) \le b_t + 2\eta \text{ for } t=1,2,\dots s \quad ; \quad &x_j \in \{0,1\} \\ &\frac{1}{n}Rx \le \frac{1}{n}p \end{aligned}$$

This implies that, for now real x,

$$\begin{aligned} &\alpha + O(\epsilon) \ge \text{Max } l_0 + l_1x_1 + l_2x_2 + \dots l_nx_n \\ b_t - 2\eta + \frac{s+k}{n} \le \frac{1}{n}x(S_t) \le b_t + 2\eta - \frac{s+k}{n} \text{ for } t=1,2,\dots s \quad ; \quad &0 \le x_j \le 1 \\ &\frac{1}{n}Rx \le \frac{1}{n}p - \frac{s+k}{n} \end{aligned}$$

because, for the linear program, there is a basic optimal solution which has at most $s+k$ fractional variables and setting them to 0 gives us an integer solution whose objective value is at least the linear program value minus $O((s+k)/\epsilon n)$ which is $O(\epsilon)$.

Now we wish to apply Theorem 5. To this end, we note that $||U||_\infty \le 1/n$; and we may use $M_2 = O(1/\epsilon n)$ and $\alpha_2 = O(1/n)$ in that theorem. Also, we will use $\gamma = O(\epsilon^2)$; note that this satisfies the required lower bound on γ in that theorem. We will also use the fact that $(s+k)/n$ is at most $\eta/2$ and $\eta \ge 2\sqrt{\gamma}n||U||_\infty$ (the last requires us to choose η not too small; indeed η equal to a large constant times ϵ will do). Thus, we get for the following event $E_3(b)$ (for one fixed b) the claimed probability bound (for a suitable choice of $\gamma \in O(\epsilon^2)$)

$$\begin{aligned} &E_3(b) : \frac{q}{n}(\alpha - l_0 + O(\epsilon)) > \text{Max} \sum_{j\in Q} l_jx_j \\ b_t - \eta \le \frac{1}{q}x(S_t \cap Q) \le b_t + \eta \quad \text{for } t=1,2,\dots s \quad ; \quad &0 \le x_j \le 1. \\ &\frac{1}{q}R^Qx^Q \le \frac{1}{n}p - \frac{s+k}{n} - O(\epsilon) \\ &\mathbf{Pr}(E_3(b)) \ge 1 - e^{-10\log(1/\epsilon)/\epsilon^2} \end{aligned}$$

Applying Lemma 6, we see that $E_1, E_2, E_3(b)$ together imply

$$E_4(b) : \alpha + O(\epsilon) > \text{Max } \tilde{l}_0 + \sum_{j \in Q} \tilde{l}_j x_j$$

$$b_t - \eta \le \frac{1}{q} x(S_t \cap Q) \le b_t + \eta \quad \text{ for } t = 1, 2, \ldots s$$

$$\frac{1}{q} R^Q x^Q \le \frac{1}{n} p - \frac{s+k}{n} - O(\epsilon) \qquad ; \quad 0 \le x_j \le 1.$$

Note that by using our hypothesis on a $gap(R, p)$ we can slacken the global constraint above to

$$\frac{1}{q} R^Q x^Q \le \frac{1}{n} p$$

The upper bound on objective function of the above Linear Programming also applies to the corresponding Integer Program. Now appealing again to Lemma 6, we get that

$$E_4(b) \Longrightarrow \tilde{g}(x) \le \alpha + O(\epsilon) \; \forall x : x_j \in \{0, 1\}, j \in Q, x \in \tilde{I}(b, \eta).$$

Letting,

$$E_4 : E_4(b) \text{ holds for all } b \in \mathcal{A}^s,$$

we then see that since the $\tilde{I}(b, \eta)$ together cover all of $\{0, 1\}^q$, under E_4, we have that

$$\tilde{g}(x) \le \alpha + O(\epsilon) \; \forall x : x_j \in \{0, 1\}, j \in Q,$$

and also we have that

$$\mathbf{Pr}(E_4) \ge 1 - (\text{ number of } b \text{ 's}) e^{-10 \log(1/\epsilon)/\epsilon^2} \ge 99/100.$$

To complete the proof of (12) (and hence the Theorem), we only need to prove now that with high probability

$$\text{Max}_{x: Rx \le p} \; g(x) \le \text{Max}_{x^Q: R^Q x^Q \le (1/n)p} \; \tilde{g}(x) + O(\epsilon)$$

The proof of the analogous statement when there are no global constraints is routine. Here, we can argue as follows. Fix an arbitrarily small positive α. By Lemma 2 we have that

$$Max_{x: Rx \le p - \epsilon \mathbf{1}} \; g(x) \ge \text{Max}_{x: Rx \le p} \; g(x) - O(\alpha)$$

By standard statistics, we can assert that the induced assignment x^Q satisfies with probability at least 9/10 (if n is sufficiently large) to the constraints $\frac{1}{q} R^Q x^Q \le \frac{1}{n} p$. We can also assert that with probability at least 1/10 we have that $\tilde{g}(x^Q) \ge (1 - O(\epsilon)) g(x)$ which implies with the last inequality,

$$\tilde{g}(x^Q) \ge (1 - O(\epsilon)) \text{Max}_{x: Rx \le p} \; g(x).$$

This concludes the proof.

5 Coping with Balanced Constraints

A important class of constraint satisfaction problems for which the global constraints have gap 0 are the so-called balanced problems of MAX-rCSP, such as the BISECTION problem for graphs or hypergraphs. Here the number of positive and the number of negated variables in the solution are bound to be equal. Let us show how these problems can in fact be approximated by a simple modification of our general algorithm. This gives also the best up to now sample complexity bound $O\tilde{}(1/\epsilon^4)$ for those problems.

We denote the number of variables by $2n$. We will prove the following theorem.

Theorem 7. *Let $r, \epsilon, s, \delta, k$, and R, be as in Theorem 1. We add to R the balance constraint:*

$$\sum_{i=1}^{2n} x_i = n$$

and we let R^* denote the augmented set of constraints. Assume that for some fixed η with $0 \le \eta \le \epsilon$, the following system of inequalities is feasible:

$$Rx \le p - \eta n$$

$$\sum_{i=1}^{2n} x_i \le n(1+\eta)$$

$$\sum_{i=1}^{2n} -x_i \le -n(1-\eta)$$

Let q, F, Q be as in Theorem 1. Then with probability at least 9/10, we have,

$$\left| \frac{n^r}{q^r} Max(F^Q) - Max(F) \right| \le \epsilon n^r$$

where $Max(F)$ is the maximum number of functions in F which can be simultaneously set to 1 by a truth assignment x to the variables satisfying the constraints $R^*x \le p$, $Max(F^Q)$ is the maximum number of functions in F^Q which can be simultaneously set to 1 by a truth assignment to the variables Q satisfying the constraints

$$\frac{1}{q} Rx^Q \le \frac{1}{n} p - \eta$$

$$\frac{q(1-\eta)}{2} \le \sum_{i \in Q} x_i \le \frac{q(1+\eta)}{2}$$

Proof Using Theorem 1 we get an approximate solution within ϵn^r, of the relaxed problem

$$Rx \le p - \eta n$$

$$n(1-\eta) \leq \sum_{i=1}^{2n} x_i \leq n(1+\eta).$$

Now, flipping the necessary number of variables in this solution to satisfy the balance condition (this number is at most ηn) gives a solution y satisfying $Ry \leq p$, i.e., a feasible solution, whereas the value of the objective function decreases by at most $\eta n^r \leq \epsilon n^r$.

□

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