TSP with Bounded Metrics:
Stronger Approximation Hardness

Lars Engebretsen[1,*] and Marek Karpinski[2,†]

[1] Department of Numerical Analysis and Computer Science
Royal Institute of Technology
SE-100 44 Stockholm
E-mail: enge@kth.se

[2] Department of Computer Science
University of Bonn
53117 Bonn
E-mail: marek@cs.uni-bonn.de

Abstract

The general asymmetric TSP with triangle inequality is known to be approximable only within logarithmic factors. In this paper we study the asymmetric and symmetric TSP problems with *bounded metrics*, i.e., metrics where the distances are integers between one and some constant upper bound. Recently, Papadimitriou and Vempala announced improved approximation hardness results for both symmetric and asymmetric TSP with graph metric. In this note, we show that a similar construction can be used to obtain only slightly weaker approximation hardness results for TSP with triangle inequality and distances that are integers between one and eight. This shows that the Papadimitriou-Vempala construction is "local" in nature and, intuitively, indicates that it cannot be used to obtain hardness factors that grow with the size of the instance.

Key words. Approximation Hardness; Metric TSP; Bounded Metric.

1 Introduction

A common special case of the Travelling Salesman Problem (TSP) is the *metric TSP*, where the distances between the cities satisfy the triangle inequality. The decision version of this special case was shown to be **NP**-complete by Karp [7],

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which means that we have little hope of computing exact solutions in polynomial time. Christofides [1] has constructed an elegant algorithm approximating the metric TSP within 3/2, i.e., an algorithm that always produces a tour whose weight is at most a factor 3/2 from the weight of the optimal tour. For the case when the distance function may be asymmetric, the best known algorithm approximates the solution within $O(\log n)$, where n is the number of cities [6]. As for lower bounds, a recent result of Engebretsen and Karpinski [4] shows that it is **NP**-hard to approximate TSP where the distances are constrained to be either one or two—note that such a distance function always satisfies the triangle inequality—within $321/320 - \varepsilon$ and that it is **NP**-hard to approximate the symmetric TSP with distances one and two within $741/740 - \varepsilon$ for every constant $\varepsilon > 0$. Papadimitriou and Vempala [8] recently announced stronger approximation hardness results for the asymmetric and symmetric versions of the TSP with graph metric, but left the case of TSP with *bounded metric* open. (Their original proof contained an error influencing the explicit constants. A new proof with the new constants $117/116 - \varepsilon$ and $220/219 - \varepsilon$, respectively, was announced by Papadimitriou and Vempala in May 2002; the latest version of the paper is available from URL http://www-math.mit.edu/~vempala/papers/tspinapprox.ps). Apart from being an interesting question on its own, it is conceivable that the special cases with bounded metric are easier to approximate than the cases when the distance between two points can grow with the number of cities in the instance. Indeed, the asymmetric TSP with distances bounded by B can be approximated within B by just picking any tour as the solution.

Definition 1. *The* Asymmetric Travelling Salesman Problem (ATSP) *is the following minimisation problem: Given a collection of cities and a matrix whose entries are interpreted as the distance from a city to another, find the shortest tour starting and ending in the same city and visiting every city exactly once.*

Definition 2. (1,B)-ATSP *is the special case of ATSP where the entries in the distance matrix obey the triangle inequality and the off-diagonal entries in the distance matrix are integers between 1 and B.* (1,B)-TSP *is the special case of (1,B)-ATSP where the distance matrix is symmetric.*

In this paper, we prove that a slight modification to the recent construction of Papadimitriou and Vempala shows that it is, for any constant $\varepsilon > 0$, **NP**-hard to approximate (1,8)-ATSP within $135/134 - \varepsilon$ (Theorem 2) and that it is, for any constant $\varepsilon > 0$, **NP**-hard to approximate (1,8)-TSP with $389/388 - \varepsilon$ (Theorem 3). In a preliminary version of this paper [3], we erroneously claimed slightly better bounds.

2 Overview of the "unbounded" construction

Papadimitriou and Vempala [9] prove their hardness result by reduction from Håstad's approximation hardness result for systems of linear equations [5].

Theorem 1 ([5]). *For any constant $\delta \in (0, 1/2)$, there exists systems of linear equations mod 2 with $2m$ equations and exactly three unknowns in each equation such that: 1) Each variable in the instance occurs a constant number of times, half of them negated and half of them unnegated. This constant grows as $\Omega(2^{1/\delta})$. 2) Either there is an assignment satisfying all but at most δm equations, or every assignment leaves at least $(1-\delta)m$ equations unsatisfied. 3) It is* ***NP****-hard to distinguish between these two cases.*

From a system of linear equations with the properties described in Theorem 1, Papadimitriou and Vempala construct an instance of ATSP by hooking several *gadgets* together. Each equation is represented by an *equation gadget* of the form shown in Fig. 1. The ticked edges in that figure in fact correspond to gadgets themselves; these gadgets are shown in Fig. 2. The construction is parameterised; Papadimitriou and Vempala set $a = 4$, $b = 2$ and $d = 6$ in the current version of their paper [9]. The main idea in the construction is that the way a TSP tour traverses the latter gadgets mentioned above, the so called *edge gadgets*, gives an assignment to the variables in the underlying system of linear equations (see Figs. 3 and 4). The main technical challenge is to prove that there is a correspondence between the length of TSP tours in the constructed graph and the number of equations satisfied by the corresponding assignment. To this end, Papadimitriou and Vempala devised a way to connect the edge gadgets corresponding to the same variable in a net with certain expander-type properties. Informally, the structure of this net is such that any attempt to construct a TSP-tour that represents the value of a certain variable inconsistently in the gadgets corresponding to the equations where that variable occurs gives a tour of high cost. Intuitively, it is therefore always suboptimal to construct such "cheating" TSP-tours.

More formally, Papadimitriou and Vempala introduces the notion of a *b-pusher* [9, Definition 1] to precisely describe the structure that is needed to thwart "cheating" TSP-tours: A d-regular bipartite graph with vertex set $V_1 \uplus V_2$ is called a *b-pusher* if, for any partition of V_1 into subsets U_1, S_1, T_1 and any partition of V_2 into subsets U_2, S_2, T_2 such that there are no edges from vertices in U_1 to vertices in U_2, the number (T_1, T_2) of edges between vertices in T_1 and T_2 satisfies

$$\left(b + \frac{1}{2}\right)(T_1, T_2) \geq \min\bigl\{|U_1| + |T_2|, |T_1| + |U_2|\bigr\} - \left(b - \frac{1}{2}\right)\bigl(|S_1| + |S_2|\bigr).$$

Papadimitriou and Vempala establish the existence of 6-regular 2-pushers [9, Theorem 5.1] and use such graphs to construct the precise coupling between different edge gadgets.

2.1 Modifications for bounded metrics

An inspection of the details of the Papadimitriou-Vempala construction shows that it, essentially, uses a metric which is bounded, in our sense of the word, by some constant that depends on ε. Qualitatively, their result is therefore

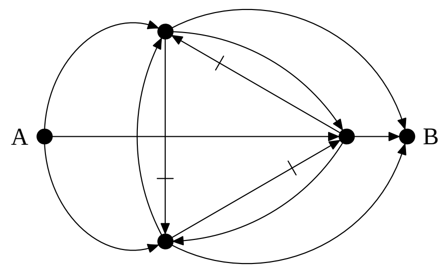

Figure 1: The gadget for equations of the form $x + y + z = 0$. There is a Hamiltonian path from A to B only if zero or two of the ticked edges, which are actually gadgets themselves (Fig. 2), are traversed. The non-ticked edges have weight 1.

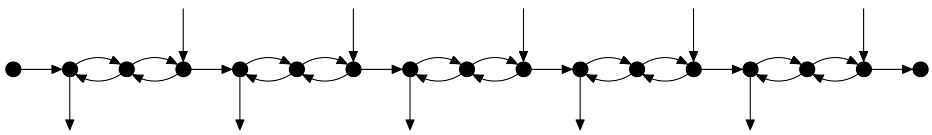

Figure 2: The edge gadget consists of d bridges. Each of the bridges are shared between two different edge gadgets. Each bridge consist of aL undirected edges of weight $1/L$ each. In the construction of Papadimitriou and Vempala [9], L is a (very) large integer constant—in our construction for bounded metrics, $L = 1$. The edges between bridges have weight b, the first horizontal edge has weight $\lfloor \frac{b+1}{2} \rfloor$, and the last horizontal edge has weight $\lceil \frac{b+1}{2} \rceil$.

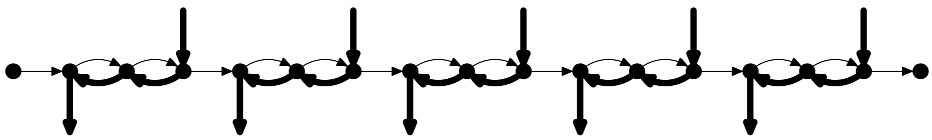

Figure 3: An untraversed edge gadget represents the value 0.

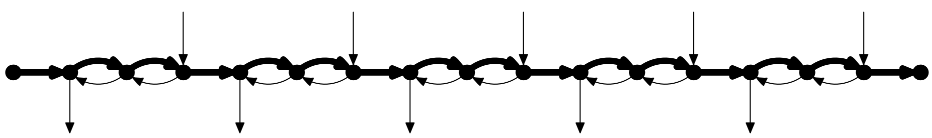

Figure 4: A traversed edge gadget represents the value 1.

of the form "there exists a constant c such that for every $\varepsilon > 0$ it is hard to approximate TSP within $c - \varepsilon$ in instances with metrics bounded by $B(\varepsilon)$". Our result in this paper is, again qualitatively, that the order of the quantifiers may be reversed, i.e., our result is of the form "there exists constants B and c such that for every $\varepsilon > 0$ it is hard to approximate TSP within $c - \varepsilon$ in instances with metrics bounded by B". Quantitatively, Papadimitriou and Vempala [9] have $c = 117/116$ for the asymmetric TSP and $c = 220/219$ for the symmetric TSP. For our case, the result is a trade-off between B and c. We settle for $B = 8$ which gives $c = 135/134$ for the asymmetric TSP and $c = 389/388$ for the symmetric TSP.

As mentioned in the caption of Fig. 2, the edge gadgets devised by Papadimitriou and Vempala [9] contain edges with very small weight. Specifically, the weight of the lightest edge in the instance is negligible compared to the constant ε in the main hardness result. In our model for bounded metrics, we only allow distances that are integers between one and some bound B. Consequently, we must modify the bridges in the edge gadgets so that they contain a edges of weight one instead of aL edges of weight $1/L$. This modification implies that the analysis must be modified. In particular, the so called "doubly traversed bridges", that incur an extra cost of $a + b$ in the Papadimitriou-Vempala construction, only incur a cost of $a + b - 2$ in our case. We believe that it is more natural to view those bridges as a kind of "semitraversed edge gadget" in our case. This change implies that a certain trick used by Papadimitriou and Vempala to associate a larger cost with the semitraversed edge gadgets does not work.

To conclude, we obtain weaker bounds on the cost incurred by "cheating TSP tours" in our case. This means that we cannot use the 6-regular 2-pushers used by Papadimitriou and Vempala—to use the straightforward reduction, we would instead need 2.5-pushers. It is easy to prove that 8-regular 2.5-pushers exist. However, using 8-regular graphs instead of 6-regular ones gives weaker approximation hardness results. To improve our results somewhat, we use a slightly more elaborate reduction, that does not need pushers but bipartite graphs with slightly weaker properties. As the final link in the proof of our hardness results, we show that there exist 7-regular graphs with the properties we need for our analysis to go through.

3 The hardness of (1,B)-ATSP

The purpose of this section is to show that the Papadimitriou-Vempala construction can be analysed also in the setting of *bounded metrics* with only small modifications. Specifically, we prove the following result:

Theorem 2. *For any sufficiently small constant $\varepsilon > 0$, there exists for any large enough integer m instances of (1,8)-ATSP with $113m$ cities such that: 1) Either there is a TSP tour with length at most $(134 + \varepsilon)m$ or else every TSP tour has length at least $(135 - \varepsilon)m$. 2) It is* ***NP****-hard to distinguish these two cases.*

The proof of this theorem follows from Lemmas 1 and 2 described below.

We describe our instance of (1,B)-ATSP by constructing a weighted directed graph and then let the (1,B)-ATSP instance have the nodes of this graph as cities. In this paper we denote by $\ell(u, v)$ the distance from u to v in this weighted graph and define the distance between two cities u and v is the (1,B)-ATSP instance, denoted by $c(u, v)$, as $c(u, v) = \min\{\ell(u, v), B\}$.

3.1 The gadgets

The gadgets are parameterised by the parameters a, b and d; they will be specified later. The equation gadget for equations of the form $x + y + z = 0$ is shown in Fig. 1. The following property of the equation gadget was established by Papadimitriou and Vempala [9]:

Proposition 1. *There is a Hamiltonian path of length four through the gadget only if zero or two of the ticked edges are traversed. All other traversals have cost at least five.*

The equation gadgets are connected in a circle by identifying vertex B in one gadget with vertex A in the next gadget in the circle.

The ticked edges in Fig. 1 are gadgets themselves. This gadget is shown in Fig. 2. Each of the bridges is shared between two different edge gadgets, one corresponding to a positive occurrence of the literal and one corresponding to a negative occurrence. The precise coupling is provided by a certain d-regular bipartite multigraph. Specifically, proceed as follows for each literal x: Let k be the number of occurrences of x (and therefore also of $\bar{x}$); Take a bipartite d-regular multigraph with vertex set $V_1 \cup V_2$ ($|V_1| = |V_2| = k$); Label the vertices in V_1 with the occurrences of x and the vertices in V_2 with the occurrences of $\bar{x}$; Let a positive and a negative occurrence correspond to the same edge gadget if there is an edge between the corresponding vertices in the bipartite graph—the order of the occurrences inside the edge gadget is not important. Later, we describe some additional required properties of the bipartite multigraph, for now it only remains to mention that they can be constructed in constant time since they are of constant size.

3.2 Constructing a tour from an assignment

Consider a system of linear equations with the properties described in Theorem 1 and an instance of (1,B)-ATSP constructed from it as described in § 3.1. Let π be an assignment to the variables in the system of linear equations and consider the tour that 1) For each variable x traverses the edge gadget corresponding to x as shown in Fig. 3 if $\pi(x) = 0$ and as shown in Fig. 4 if $\pi(x) = 1$. 2) For each equation gadget enters each equation gadget at node A, takes the shortest possible way to B under the condition that the ticked edges are traversed as prescribed by the traversals of the edge gadgets, and then exits the equation gadget at node B.

Since there are $2m$ equations in the system of linear equations, the number of cities contained in the equation gadgets is $4 \cdot 2m = 8m$. Similarly, since every edge gadget is shared between two equation gadgets, there are $2m \cdot \frac{3}{2}d(a+1) = 3md(a+1)$ cities inside the equation gadget.

The length of the tour described above "inside" the edge gadgets is $d(a+b)$. The "extra" cost of one that comes from the two "outermost" horizontal edges in Fig. 2 is attributed to the equation gadget; in this way we can assign a cost of one to all edges in Fig. 1. Since there are $2m$ equations, three edge gadgets per equation gadget, and every edge gadget is shared between two equation gadgets, it follows that the total cost of the tour inside the edge gadgets is $3md(a+b)$. Considering an arbitrary equation gadget, the path from A to B in a tour constructed as described above has length four if the corresponding equation in the system of linear equations is satisfied by the assignment π and length five otherwise. (Strictly speaking, it is impossible to have three traversed edge gadgets in an equation gadget, since this does not result in a TSP tour. However, we can regard the case when the tour of the third edge gadget leaves the edge gadget by jumping directly to the exit node of the equation gadget as a tour with three traversals; such a tour gives a cost of five, in addition to the cost attributed to the edge gadgets.) Hence, the total cost accounted to the equation gadgets is $8m+u$, where u is the number of unsatisfied equations. We summarise the above discussion:

Lemma 1. *Consider a system of linear equations with the properties described in Theorem 1 and an instance of (1,B)-ATSP constructed from it as described in § 3.1. This instance contains $3md(a+1)+8m$ cities. Given an assignment to the variables in the system of linear equations that satisfies all but u equations, the tour produced from this assignment as described above has length $3md(a+b)+8m+u$.*

3.3 Constructing an assignment from a tour

The main challenge now is to prove that the above correspondence between the length of the optimum tour and the number of unsatisfied equation holds also when we drop the assumption that the tour is shaped in the intended way. Specifically, the aim is to show the following:

Lemma 2. *Consider a system of linear equations with the properties described in Theorem 1 with δ sufficiently small and an instance of (1,B)-ATSP constructed from it as described in § 3.1 with $a=4$, $b=2$, $d=7$, and $B=8$. Any TSP tour of length $3md(a+b)+8m+u$ in this instance can be used to construct in polynomial time an assignment satisfying all but at most u equations.*

Our proof uses three technical lemmas. The first one shows that any tour can be transformed into a tour with a certain behaviour inside the bridges. The second lemma lower bounds the additional cost caused by non-standard traversals of an edge gadget and the last lemma establishes that the bipartite graph used has a certain expansion-related property.

Lemma 3. *Consider a system of linear equations with the properties described in Theorem 1 and an instance of (1,B)-ATSP constructed from it as described in § 3.1. If $B \geq a$, any TSP tour in such an instance can be transformed in polynomial time into a tour with smaller, or equal, length with the following properties:*

1) Let (u, v) be an edge of the tour and suppose that u and v both belong to the same bridge. Then u and v are neighbours in the graph defining the (1,B)-ATSP instance.

2) Let u and v be neighbours on the same bridge and assume that there is no edge between u and v in the tour. Let (u, u') and (v, v') be edges of the tour and assume that $c(u, u') = \ell(u, u')$ and that $c(v, v') = \ell(v, v')$. Then the shortest path from u to u' does not intersect the shortest path from v to v'.

Definition 3. *A bridge has a* defined traversal *if the tour restricted to the bridge is a path of length a; otherwise the bridge has an* undefined traversal*.*

Definition 4. *An edge gadget is* traversed *if all bridges have defined traversals and the connection edges (horizontal in Fig. 2) are traversed by the tour; it is* untraversed *if all bridges have defined traversals and none of the the connection edges are traversed by the tour. All other edge gadgets are* semitraversed*.*

Lemma 4. *Consider a system of linear equations with the properties described in Theorem 1 and an instance of (1,B)-ATSP constructed from it as described in § 3.1. From a tour with the properties guaranteed by Lemma 3, it is possible to associate a cost of at least $\min\{a/2, b, a/2 + b/2 - 1\}$ with every semitraversed edge gadget given that $B \geq \max\{3b, a + b, 2a + b - 2\}$.*

Lemma 5. *For every large enough constant k, there exists a 7-regular bipartite multigraph with vertex set $V_1 \uplus V_2$ ($|V_1| = |V_2| = k$) such that for every partition of V_1 into sets T_1, U_1 and S_1 and every partition of V_2 into sets T_2, U_2 and S_2 such that there are no edges from T_1 to T_2, and there are no edges from U_1 to U_2,*

$$2\big(|S_1| + |S_2|\big) \geq \min\big\{k, |U_1| + |T_2| + |S_1| + |S_2|, |U_2| + |T_1| + |S_1| + |S_2|\big\}.$$

Before proving these lemmas, we show that they give—by appropriate choice of parameters—the desired connection between the length of an arbitrary TSP tour and the number of satisfied equations in the corresponding system of linear equations.

Proof of Lemma 2. Set $a = 4$, $b = 2$, $d = 7$, and $B = 8$. Then it follows from Lemma 4 that every semitraversed edge gadget incurs a cost of at least two.

For every variable x, let the bipartite multigraph used to construct the edge gadget have the property stated in Lemma 5 with k equal to the number of occurrences of x (and hence also of $\bar{x}$). Lemma 5 asserts that such graphs exist for sufficiently large k; hence we must assume that δ in Theorem 1 is small enough.

The assignment to an arbitrary variable x is constructed as follows: Suppose that x occurs k times positively and k times negatively. Let T_1 be the set of traversed positive occurrences and T_2 be the set of traversed negative occurrences. Define U_1, U_2, S_1, and S_2 similarly. If $|S_1| + |S_2| \geq k/2$, set $\pi(x) = 0$ with probability $1/2$ and $\pi(x) = 1$ with probability $1/2$. Otherwise define $\pi(x)$ deterministically as follows: If $|T_1| + |U_2| \geq |T_2| + |U_1|$, let $\pi(x) = 1$, otherwise let $\pi(x) = 0$. The resulting probabilistic assignment is then derandomised, using the method of conditional probabilities, to produce an assignment satisfying at least as many equations as the expected number of equations satisfied by π.

We need to prove that there is at most one unsatisfied equation per unit of the "extra" cost u, i.e., per unit of the cost in addition to the "normal" cost of $3md(a+b)$ for the edge gadgets and $8m$ for the equation gadgets. To this end, we show that it is possible to associate a cost of at least $1/2$ with every equation containing a variable that has been set at random and a cost of at least 1 with every other equation that could be unsatisfied by π.

Let x be an arbitrary variable and suppose that x occurs $2k$ times. Define T_1, T_2, U_1, U_2, S_1, and S_2 as above. Since variables are given probabilistic assignments only when $|S_1| + |S_2| \geq k/2$ and every semitraversed edge gadgets incurs an extra cost of 2, there is an extra cost of at least $1/2$ associated with every equation containing a variable that has been assigned a random value. Since every such eqation is satisfied with probability $1/2$, no matter the number of variables in the equation that were given random assignments, the extra cost attributed to variables with a random assignment is equal to the expected number of unsatisfied equations from this assignment.

Consider next the case when $|S_1| + |S_2| \leq k/2$. Since Lemma 5 guarantees that the extra cost incurred by the semitraversed occurrences of x and $\bar{x}$ is no less than

$$\min\bigl\{|U_1| + |T_2| + |S_1| + |S_2|, |U_2| + |T_1| + |S_1| + |S_2|\bigr\}$$

in this case, the extra cost incurred by the semitraversed occurrences pays for the potential unsatisfaction of every equation that contains a variable that has been assigned a value contradicting the traversal of the corresponding edge gadget. The only remaining possibility for equations that are unsatisfied under π comes from equations where all variables have been assigned values according to the traversal of the corresponding edge gadgets and that assignment does not satisfy the equation. However, for such equations, there is an extra cost of one in the equation gadget according to Proposition 1. □

3.4 Proof of Lemma 3

To ensure property 1, repeat the following for all edges (u, v) of the tour such that u and v belong to the same bridge but are not neighbours in the graph: Redefine the tour, so that instead of jumping from u directly to v, the tour follows the shortest path from u to v in the graph defining the instance. Since $B \geq a$ this does not increase the length of the tour. This change will make the tour pass through some cities—the cities that are on the shortest path from u to v in the graph—twice. For all such cities w, do the following: Let w' be the city visited immediately before w and w'' be the city visited immediately after w. Then replace the edges (w', w) and (w, w'') by the single edge (w', w'') in the tour. By triangle inequality this procedure does not increase the length of the tour.

To ensure property 2, repeat the following for all vertices u and v that belong to the same bridge but for which there is no edge between u and v in the tour: Let u' and v' be defined as in the formulation of the lemma. If the shortest path from u to u' does not intersect the shortest path from v to v', no transformation of the tour is needed. Otherwise, the fact that u and v are on the same bridge implies that we can assume without loss of generality that the shortest path from u to u' passes v (otherwise we just exchange u and v in the argument). We then redefine the tour, so that instead of jumping from u directly to u', the tour follows the shortest path from u to u' in the graph defining the instance. As above, for every node w on the shortest path from u to u' (including v), let w' be the city visited immediately before w and w'' be the city visited immediately after w and replace the edges (w', w) and (w, w'') by the single edge (w', w'') in the tour. By triangle inequality this procedure does not increase the length of the tour.

3.5 Proof of Lemma 4

Consider a semitraversed edge gadget. We now argue by case analysis that it introduces an extra cost in addition to the "standard" cost of $a + b$ per bridge. For accounting purposes, we use the convention that this standard cost corresponds to a cost of $b/2$ for the incoming edge of the tour plus a cost of $b/2$ for the outgoing edge of the tour plus a cost of a for the traversal of the bridge itself. When analysing the extra cost due to semitraversals, it is important to attribute this extra cost to both edge gadgets that take part in the semitraversal. Sometimes this means two different edge gadgets that represent the same literal x (or $\bar{x}$); sometimes this means the two edge gadgets that cross at a certain bridge. For "long" jumps, i.e., cases when the tour traverses an edge (u, v) with cost $c(u, v) \neq \ell(u, v)$, a cost of $B/2$ is attributed to both of the involved bridges.

Lemma 6. *Given that $B \geq 2a + b - 2$, it is possible to associate a cost of at least of at least $a/2 + b/2 - 1$ with every edge gadget that becomes semitraversed because of a bridge having an undefined traversal.*

Proof. We first consider the case when the metric is not bounded; we will show later how to extend the argument to cover also bounded metrics. In the unbounded case, the distance between two vertices u and v is exactly the length of the shortest path from u to v in the graph defining the instance.

Since the bridge has an undefined traversal, there must be two adjacent cities u and v that are not neighbours in the tour. Consider the edges (u, u')

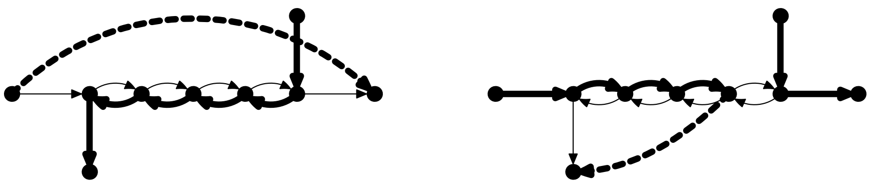

Figure 5: We can assume that traversals shown in the left figure above never occur since they can be transformed into the traversal shown in the right figure without increasing the length of the tour. A bridge with a traversal of that form gives an extra cost of at least $\min\{a+b-2, a+b/2-1\}$ if $B \geq 2a+b-2$.

and (v, v') in the tour—thanks to Lemma 3 we can assume that neither u' nor v' belong to the bridge.

The tour must visit all cities on the bridge. Therefore the total cost of the tour on the bridge is, according to our convention, at least $2a+2b-2$, which gives an extra cost of $a+b-2$.

When the metric is bounded by some bound B, a case analysis shows, that if $B/2 \geq a+b/2-1$ it follows that the cost of the tour on a bridge with an undefined traversal is still at least $2a+2b-2$. Intuitively, this states that the case shown to the right in Fig. 5 with the dotted line replaced by a "jump" following some edge with cost B is the worst case, i.e., the case with lowest extra cost.

Since a bridge containing an undefined traversal makes both edge gadgets passing through it semitraversed, the proof of the lemma is complete. □

Lemma 7. *Given that* $B \geq \max\{a+b, 3b\}$ *it is possible to associate a cost of at least* $\min\{a/2, b\}$ *with every edge gadget that becomes semitraversed because of a bridge with a defined traversal.*

Proof. We first consider the case when the metric is not bounded and show later how to extend the argument to cover also bounded metrics. In the unbounded case, the distance between two vertices u and v is exactly the length of the shortest path from u to v in the graph defining the instance.

Consider first a bridge traversed from left to right but where the connecting edge leaving the bridge is not traversed by the tour. Hence, the tour makes a jump leaving the bridge. There are three sub-cases:

The tour goes down (Fig. 6). The earliest available free city is a distance of $2b$ away; that blocks the tour leaving the right bridge, forcing it to also make a jump of at least $2b$. The next available free city is a distance of $3b$ away. Both these cases give a total extra cost of $2b$.

The tour goes forwards (Fig. 7). The earliest available free city is a distance of $a+b$ away, giving a total extra cost of a.

The tour goes backwards (Fig. 8). The earliest available free city is a distance of $a+b$ away, giving a total extra cost of a.

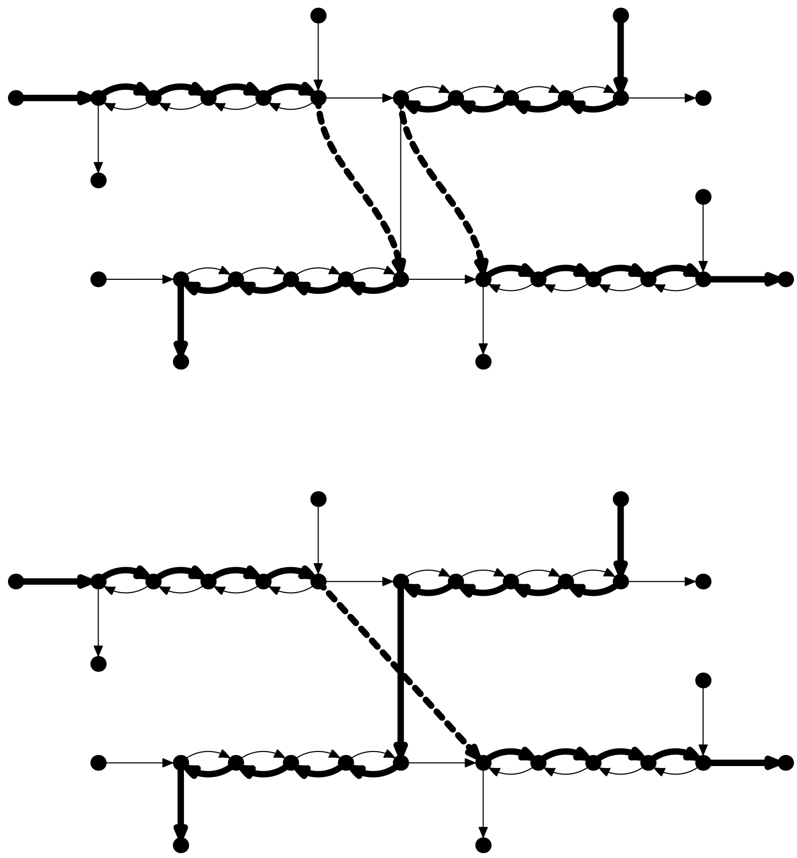

Figure 6: Switching from traversing an edge gadget representing an occurrence of x to traversing another edge gadget representing an occurrence of x gives an extra cost of at least b. The dotted edge above has length $3b$; that gives an extra cost of $2b$ which is then shared evenly among the two semitraversed edge gadgets.

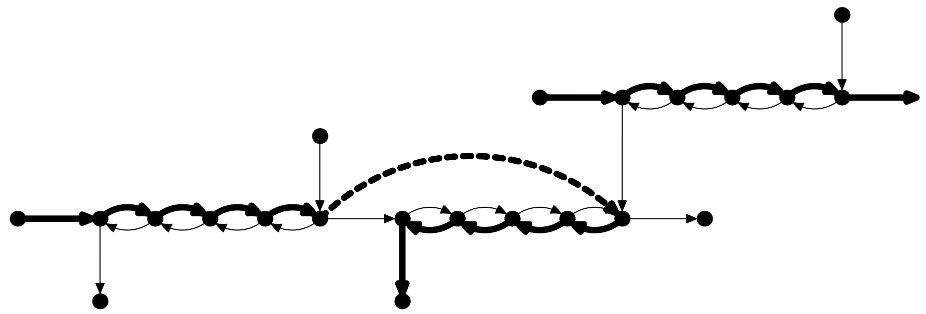

Figure 7: Switching from traversing an edge gadget representing an occurrence of x to traversing an edge gadget representing an occurrence of $\bar{x}$ gives an extra cost of at least $a/2$. The dashed edges above has length $a + b$; that gives an extra cost of a which is then shared evenly among the two semitraversed edge gadgets.

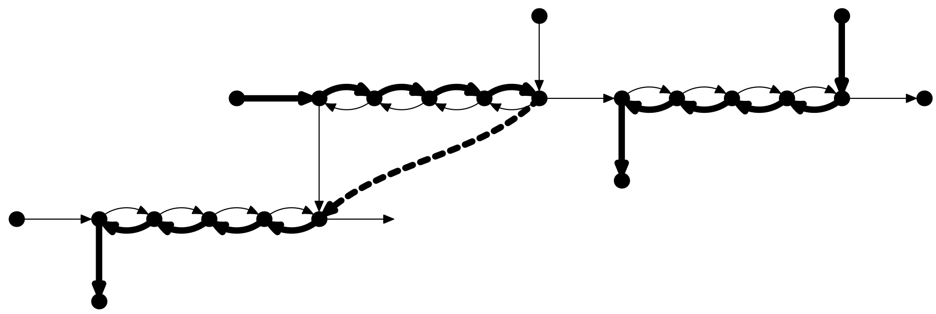

Figure 8: Switching from traversing an edge gadget representing an occurrence of x to traversing an edge gadget representing an occurrence of $\bar{x}$ gives an extra cost of at least $a/2$. The dashed edges above has length $a + b$; that gives an extra cost of a which is then shared evenly among the two semitraversed edge gadgets.

Next, consider a bridge traversed from left to right where the connecting edge *entering* the bridge is not traversed by the tour. Again, there are three sub-cases.

The tour comes from above (Fig. 6). The earliest available free city is a distance of $2b$ away, but that blocks the tour entering the right bridge, forcing it to also make a jump of at least $2b$. The next available free city is a distance of $3b$ away. Both these cases give a total extra cost of $2b$.

The tour comes from the front (Fig. 7). The earliest available free city is a distance of $a + b$ away, giving a total extra cost of a.

The tour comes from behind (Fig. 8). The earliest available free city is a distance of $a + b$ away, giving a total extra cost of a.

So far, the analysis only considered unbounded metrics. Note first, however, that if $B \geq \max\{3b, a + b\}$, the above argument is valid. If the tour makes a larger jump than the shortest possible jumps stated above, the additional cost can never decrease, thanks to the triangle inequality. Next, note that if the tour leaves a bridge with a defined traversal with a "long jump", i.e., following an edge (u, v) where $c(u, v) \neq \ell(u, v)$, that particular bridge can only cause one of the edge gadgets passing through it to be semitraversed and hence we can allocate the entire net cost of $B/2 - b/2$ to that edge gadget. If $B \geq \max\{3b, a + b\}$, then $B/2 - b/2 \geq \max\{a/2, b\}$, hence the lemma holds also in this case. $\square$

Note, finally, that the above analysis is valid also for tours such that a "long jump" may start in a semitraversed gadget with no undefined traversal and end in an undefined traversal, and vice versa.

3.6 Proof of Lemma 5

The proof uses the same main idea as the proof that establishes existence of 6-regular 2-pushers: It uses the fact that it is possible to lower bound the size of neighbours to any given set of vertices in d-regular bipartite graphs. For a set W, let $N(W)$ denote the neighbours of W in the graph. With this notation, a recent study of Engebretsen [2] implies that there exist, for every large enough k, a 7-regular bipartite multigraph with vertex set $V_1 \cup V_2$ ($|V_1| = |V_2| = k$) such that for every $W \subseteq V_1$ and every $W \subseteq V_1$, the following holds:

$$
\begin{aligned}
&|W| \leq 0.15k \implies |N(W)| > 8|W|/3, \\
&0.15k \leq |W| \leq 0.60k \implies |N(W)| > 0.25k + |W|, \\
&|W| \geq 0.60k \implies |N(W)| > 5k/8 + 3|W|/8, \\
&|W| \leq 0.31k \implies |N(W)| > 2|W|, \\
&0.31k \leq |W| \leq 0.35k \implies |N(W)| > 0.31k + |W|, \\
&|W| \geq 0.35k \implies |N(W)| > 31k/65 + 34|W|/65.
\end{aligned}
$$

Our task is to prove that for every partition of the left vertices into sets T_1, U_1 and S_1 and every partition of the right vertices into sets T_2, U_2 and S_2 such that there are no edges from T_1 to T_2, and there are no edges from U_1 to U_2,

$$2\big(|S_1|+|S_2|\big) \geq \min\big\{k, |U_1|+|T_2|+|S_1|+|S_2|, |U_2|+|T_1|+|S_1|+|S_2|\big\}.$$

Since there are no edges between T_1 and T_2 and there are no edges between U_1 and U_2, it follows that $|S_1| \geq |N(T_2)| - |U_1|$. Similarly, $|S_2| \geq |N(U_1)| - |T_2|$. Also, it is easy to see that $|T_1| \leq k - |N(T_2)|$ and that $|U_2| \leq k - |N(U_1)|$. These observations are used repeatedly in the case analysis below.

From now on, we use the shorthands $|T_1| = kt_1$, $|U_1| = ku_1$, $|S_1| = ks_1$, $|T_2| = kt_2$, $|U_2| = ku_2$, and $|S_2| = ks_2$. We can also assume without loss of generality that $u_1 + t_2 \leq t_1 + u_2$. Hence, we must show that

$$2s_1 + 2s_2 \geq \min\{1, u_1 + t_2 + s_1 + s_2\}. \tag{1}$$

We let $n(x)$ denote the size of the neighbours of some set with size x. The following, somewhat overlapping, cases cover all possible values of u_1 and t_2. Hence, they are enough to complete the proof of the lemma.

Case I: $u_1 \leq 0.31$ **and** $t_2 \leq 0.31$. In this case $s_1+s_2 \geq n(t_2)-u_1+n(u_1)-t_2 \geq u_1 + t_2$, which implies (1).

Case II: $0.15 \leq u_1 \leq 0.60$ **and** $0.15 \leq t_2 \leq 0.60$. Since $s_1 \geq n(t_2) - u_1 \geq t_2 + \frac{1}{4} - u_1$ and $s_2 \geq n(u_1) - t_2 \geq u_1 + \frac{1}{4} - t_2$ in this case, it follows that $s_1 + s_2 \geq \frac{1}{2}$, which implies (1).

Case III: $u_1 \geq 0.35$ **and** $t_2 \geq 0.35$. Using the fact that $u_1 + t_2 \leq t_1 + u_2 \leq 2 - n(t_2) - n(u_1) \leq \frac{68}{65} - \frac{34}{65}u_1 - \frac{34}{35}t_2$, or, equivalently, that $u_1 + t_2 \leq \frac{68}{99} < 0.70$, we reach a contradiction since $u_1 + t_2$ must be at least 0.70 in this case. Hence this case cannot occur.

Case IV a: $u_1 \leq 0.35$ **and** $t_2 \geq 0.60$. In this case $s_1 \geq n(t_2) - u_1 \geq \frac{3}{8}t_2 + \frac{5}{8} - u_1 \geq \frac{3}{8}\cdot\frac{3}{5} + \frac{5}{8} - \frac{7}{20} = \frac{1}{2}$, which implies (1).

Case IV b: $u_1 \geq 0.60$ **and** $t_2 \leq 0.35$. In this case $s_2 \geq n(u_1) - t_2 \geq \frac{3}{8}u_1 + \frac{5}{8} - t_2 \geq \frac{3}{8}\cdot\frac{3}{5} + \frac{5}{8} - \frac{7}{20} = \frac{1}{2}$, which implies (1).

Case V a: $u_1 \leq 0.15$ **and** $t_2 \geq 0.35$. In this case $s_1 \geq n(t_2) - u_1 \geq \frac{31}{65} + \frac{34}{65}t_2 - u_1 \geq \frac{31}{65} + \frac{34}{65}\cdot\frac{35}{100} - \frac{15}{100} = \frac{51}{100} > \frac{1}{2}$, which implies (1).

Case V b: $u_1 \geq 0.35$ **and** $t_2 \leq 0.15$. In this case $s_2 \geq n(u_1) - t_2 \geq \frac{31}{65} + \frac{34}{65}u_1 - t_2 \geq \frac{31}{65} + \frac{34}{65}\cdot\frac{35}{100} - \frac{15}{100} = \frac{51}{100} > \frac{1}{2}$, which implies (1).

Case VI a: $u_1 \leq 0.15$ **and** $0.31 \leq t_2 \leq 0.35$. In this case $s_1 \geq n(t_2) - u_1 \geq t_2 + 0.31 - u_1$ and $s_2 > \max\{n(u_1) - t_2, 0\} > \max\{\frac{8}{3}u_1 - t_2, 0\}$. This gives two sub-cases that together imply (1).

$\boldsymbol{t_2 \geq \frac{8}{3}u_1}$**:** $s_1 + s_2 \geq s_1 \geq \frac{5}{8}t_2 + 0.31 \geq \frac{5}{8}\cdot 0.31 + 0.31 = \frac{403}{800} > \frac{1}{2}$.

$\boldsymbol{t_2 \leq \frac{8}{3}u_1}$**:** $s_1 + s_2 \geq \frac{5}{3}u_1 + 0.31 \geq \frac{5}{8}t_2 + 0.31 > \frac{1}{2}$.

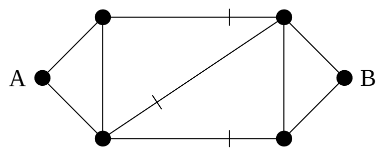

Figure 9: The symmetric gadget for equations of the form $x+y+z=1$. There is a Hamiltonian path from A to B only if an odd number of the ticked edges are traversed.

Case VIb: $0.31 \leq u_1 \leq 0.35$ **and** $t_2 \leq 0.15$. In this case $s_1 > \max\{n(t_2) - u_1, 0\} > \max\{\frac{8}{3}t_2 - u_1, 0\}$ and $s_2 \geq n(u_1) - t_2 \geq u_1 + 0.31 - t_2$. This gives two sub-cases that together imply (1).

$\boldsymbol{u_1 \geq \frac{8}{3}t_2}$**:** $s_1 + s_2 \geq s_2 \geq \frac{5}{8}u_1 + 0.31 \geq \frac{5}{8} \cdot 0.31 + 0.31 = \frac{403}{800} > \frac{1}{2}$.

$\boldsymbol{u_1 \leq \frac{8}{3}t_2}$**:** $s_1 + s_2 \geq \frac{5}{3}t_2 + 0.31 \geq \frac{5}{8}u_1 + 0.31 > \frac{1}{2}$.

4 The hardness of (1,B)-TSP

To adapt the construction from the § 3 to the symmetric case we change the gadgets; on a high level both the construction and the proof of correctness are as in the asymmetric case. The equation gadget is replaced with the gadget in Fig. 9; this gadget tests odd instead of even parity.

Proposition 2. *The only way to traverse the equation gadget in Fig. 9 with a tour of length five—if the edge gadgets count as length one—is to traverse an odd number of edge gadgets. All other traversals have length at least six.*

To construct a symmetric edge gadget, note that already the asymmetric edge gadget is in fact almost symmetric since the bridge in the asymmetric edge gadget is an undirected path of length a. Consider the following attempt to make an undirected edge gadget: Let the edges connecting the bridge with other bridges in the asymmetric edge gadget be undirected and connect the edge gadgets as in the asymmetric case. The resulting gadget penalises many, but not all, unwanted tours. In particular, the weakness with the above construction is that a path may, without any additional penalty, enter a bridge through an edge that is directed *towards* the bridge in the asymmetric version of the gadget and leave the same bridge along the *other* edge that is directed towards the bridge. To overcome this problem, we construct a symmetric version of the asymmetric bridge by hooking up three copies of the "symmetrised asymmetric bridge" described above in parallel and then rotating the resulting package 90° (see Fig. 10). We call the resulting structure a *symmetric bridge*.

Similar to the asymmetric case, we say that a symmetric bridge has a *defined traversal* if the tour restricted to the bridge traverses all three bridges and exactly two of the horizontal edges in Fig. 10. With $a = 4$, $b = 2$ and $B = 8$, the technical lemmas from § 3.5 can be used to show that any undefined traversal

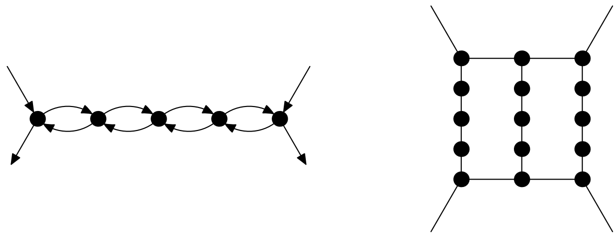

Figure 10: To transform the edge gadget from Fig. 2 into a gadget that can be used in the symmetric case, all occurrences of the structure to the left above are replaced with the structure to the right above. All vertical edges in the right figure have weight 1 and there are a edges in each of the three vertical paths; the other edges in the right figure have weight b.

of the edge gadget gives an additional local cost of four, i.e., an additional local cost of two can be attributed to each of the two edge gadgets that meet at the symmetric bridge. Defining *traversed*, *untraversed* and *semitraversed* edge gadgets as in the asymmetric case, a case analysis similar to that in the proof of Lemma 7 then shows that a cost of at least two can be associated with each semitraversed symmetric edge gadget. As in the asymmetric case, the individual edge gadgets corresponding to the same variable are stitched together according to the edges in a d-regular bipartite multigraph with vertex set $V_1 \uplus V_2$ (where $|V_1| = |V_2| = k$ and $2k$ is the number of occurrences of the variable) that has the property that for every partition of V_1 into sets T_1, U_1 and S_1 and every partition of V_2 into sets T_2, U_2 and S_2 such that there are no edges from T_1 to T_2, and there are no edges from U_1 to U_2, it holds that

$$2\bigl(|S_1| + |S_2|\bigr) \geq \min\bigl\{k, |U_1| + |T_2| + |S_1| + |S_2|, |U_2| + |T_1| + |S_1| + |S_2|\bigr\}.$$

To summarise, the following lemma follows in the same way as in the asymmetric case:

Lemma 8. *Consider a system of linear equations with the properties described in Theorem 1 with δ sufficiently small and an instance of (1,B)-TSP constructed from it as outlined above with $a = 4$, $b = 2$, $d = 7$, and $B = 8$. A TSP tour of length $9md(a + b) + 10m + u$ in this instance can be used to construct in polynomial time an assignment satisfying all but at most u equations.*

For the symmetric analogue of Lemma 1, note that a "jump" past an edge gadget actually requires following an edge of length $9md(a+b)+1$ as the construction is described above. However, by adding for every edge gadget an edge of length two that is parallel with the edge gadget in the graph defining the TSP instance, it is easy to see that the following lemma holds:

Lemma 9. *Consider a system of linear equations with the properties described in Theorem 1 and an instance of (1,B)-TSP constructed from it as outlined* *above. Given an assignment to the variables in the system of linear equations that satisfies all but u equations, it is possible to construct a TSP tour with length* $9md(a+b)+10m+u$.

Given the above lemmas, our second main theorem follows in exactly the same way as in the asymmetric case.

Theorem 3. *For any constant* $\varepsilon > 0$*, it is* ***NP****-hard to approximate (1,8)-TSP within* $389/388 - \varepsilon$.

5 Concluding remarks

There are two main conclusions from the work presented in this paper. First, the fact that it is relatively straightforward to adapt the construction devised by Papadimitriou and Vempala [9] to the case of bounded metrics shows that this latter construction is essentially local, in spite of the fact that it uses as a critical component edges with unbounded—but constant—length. This indicates that new ideas are needed to obtain hardness within factors that are $\omega(1)$, or even hardness within an arbitrarily large constant factor.

The second main conclusion is that simpler constructions and simpler proofs of correctness are needed in order to obtain hardness results that are substantially better than the currently best known ones. Current techniques have been pushed more or less to their limits. Also, earlier versions of this paper as well as earlier versions of [9] contained errors in the accounting of penalties due to non-standard traversals. In order to achieve stronger hardness results, some kind of more structured approach is probably necessary—more complicated gadget reductions and accounting procedures are bound to be even more sensitive to errors in the analysis than the construction of Papadimitriou and Vempala [9]. We believe that a direct PCP construction is the natural next step for constructing stronger approximation hardness results for TSP with triangle inequality.

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