Approximability of Hypergraph Minimum Bisection

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Abstract

We prove that the problems of minimum bisection on k-uniform hypergraphs are almost exactly as hard to approximate, up to the factor $k/3$, as the problem of minimum bisection on graphs. On a positive side, our argument gives also the first approximation algorithms for the problem of minimum bisection on k-uniform hypergraphs, for every integer k, of a comparable guarantee as for the minimum bisection on graphs. Moreover, we prove that the problems of minimum bisection on *very sparse 2-regular k-uniform hypergraphs* are precisely as hard to approximate as the general minimum bisection problem on arbitrary graphs for every integer $k \geq 3$.

1 Introduction

Hypergraph minimum partitioning and bisection problems and some heuristics for these problems became to have important applications in several problems like task scheduling, machine vision, design of integrated circuits, and databases, see cf. [KK99] and the references thereof. Despite their importance, and the fact that they are all NP-hard in exact setting [L73], very

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little is known about their approximation complexity status, and in particular, about an existence of *acceptable* approximation algorithms within the provable approximation ratios.

The problem of approximation hardness of minimum bisection on uniform hypergraphs was raised in [H02] (see also [HK03]), in a context of some generalizations of the problem of minimum bisection on graphs. The research in [H02] and [HK03] was also motivated by our current inability of proving lower approximation bounds for that latter problem, see also [K02].

In this paper we tie for the first time the approximation complexity of minimum bisection of k-uniform hypergraphs, for every fixed integer k, to the approximation complexity of minimum bisection on graphs, showing the existence of approximation algorithms for the later problem of a comparable quality with approximation algorithms for the minimum graph bisection.

2 Preliminaries

Given a *k-uniform* hypergraph $G = (V, E)$ with a set of hyperedges $E \subseteq \{h | h \subseteq V \text{ and } |h| = k\}$. For a set $U \subseteq V$ we denote by $Cut(U) = \{h | h \in E,\ h \cap U \neq \varnothing \text{ and } h \setminus U \neq \varnothing\}$ the set of hyperedges which are cut by U; we also define $cut(U) = |Cut(U)|$, and if $|U| = |V|/2$ we say that U is a bisection of G.

The *minimum bisection problem* on a k-uniform hypergraph G is the problem of constructing a bisection B of G so as to minimize the number of hyperedges which are cut by B, i.e. the number $cut(B)$. For a given integer k, we will denote this problem by *MIN-Hk-BISECTION*. The minimum bisection problem on graphs (*MIN-H2-BISECTION*) will be denoted by *MIN-BISECTION*. We will also consider a weighted version of MIN-BISECTION, where the edges of the input graph have arbitrary nonnegative weights $w : E \to \mathbb{R}_+$ and we minimize the sum of the weights of edges $w(Cut(B))$ which are cut by B.

It is well known that for every k, *MIN-Hk-BISECTION* problem is *NP-hard* in exact setting (cf. [L73]). The approximation status of MIN-Hk-BISECTION remained open for all $k \geq 3$. For the case of $k = 2$, although no approximation hardness results are currently known (cf. [BK01], [K02]), an $O(\log^2 n)$ approximation algorithm was recently designed by Feige and Krauthgamer [FK00]. Recently, Feige [F02] also proved a relative approximation lower bound of 4/3 for that problem under a hypothesis that 3SAT is hard to approximate on average. Special cases of minimum bisection on dense as well as metric graphs are known to have PTASs [AKK95], [FKK02].

This paper is concerned with the approximation hardness of MIN-Hk-BISECTION problems relative to the MIN-BISECTION problem.

For a given two functions $r : \mathbb{N} \to \mathbb{R}_+$, $t : \mathbb{N} \to \mathbb{N}$, we call an *approximation algorithm* A for an optimization problem P, an $(r(n), t(n))$-approximation algorithm for P, if A approximates P to within an approximation ratio $r(n)$ and A runs in $O(t(n))$ time for n an instance size.

3 Main Result

We are going to prove now our main result connecting closely approximation complexity of MIN-Hk-BISECTION to that of MIN-BISECTION for every integer $k \geq 3$.

Theorem 1 *If there exists an $(r(n), t(n))$-approximation algorithm for MIN-BISECTION, then there exists a $\left(\frac{k}{3}r(n^2 \log n), t(n^2 \log n)\right)$-approximation algorithm for MIN-Hk-BISECTION for any $k \geq 3$.*

Proof. Given a hypergraph $G = (V, E)$ we construct a complete weighted graph $G' = (V, E')$ where the weight of an edge $e = \{u, v\}$ is the number of hyperedges in which e is contained. Consider a possible bisection $B \subseteq V$. A hyperedge of G that is cut by B has k nodes and thus it contributes between $k - 1$ and $\lfloor k/2 \rfloor \lceil k/2 \rceil$ to the value of the cut B in G'. Therefore, if we find a bisection B in G' that has cut value r times larger than the optimum, this B has cut value in G that is at most $\frac{\lfloor k/2 \rfloor \lceil k/2 \rceil}{k-1} r$ times larger then the optimum. If k is odd then $\frac{\lfloor k/2 \rfloor \lceil k/2 \rceil}{k-1} = \frac{k+1}{4} \leq \frac{k}{3}$, and if k is even, then $\frac{\lfloor k/2 \rfloor \lceil k/2 \rceil}{k-1} = \frac{k+1}{4} + \frac{1}{4(k-1)} \leq \frac{k}{3}$.

We will reduce now the problem of MIN-BISECTION in a weighted graph with n nodes to a similar problem in an unweighted graph with $O(n^2 \log n)$ nodes using a similar approach to that of Feige and Krauthgamer [FK00]. In the latter paper the reduction produces about n^6 nodes, so we describe a more efficient reduction in some detail.

Feige and Krauthgammer [FK00] have constructed an approximation algorithm for MIN-BISECTION without edge weights with approximation ratio $(c \log n)^2$ for some constant c. Our reduction will use this algorithm as a subroutine.

We first obtain a very rough estimate of the value of minimum bisection. To do it, we start with an empty edge set E' and then we insert to it edges of E in an order of non-increasing weights. After each edge insertion we can check if the connected components of (V, E') can be grouped into two sets, each with $n/2$ nodes. Suppose that it becomes impossible after an insertion of an edge with weight α. Then each bisection is cut by an edge with a weight α or more, and there exists a bisection that is cut only by edges of cost α or less. Thus the minimum cut of a bisection, say β^*, satisfies $\alpha \leq \beta^* < \alpha n^2/4$. Therefore we can try $2\log n$ different values for β and one of them will satisfy $\beta \leq \beta^* < 2\beta$. Our later analysis will be based on the assumption that we deal with a "correct" β.

We rescale the weights so that $\beta = n^2$. Next, we round down all the weights. The latter does not increase the cost of the optimum solution, and if an algorithm finds a solution with cost γ, the true cost of this solution is below $\gamma + n^2/4$. Therefore the rounding can increase the approximation ratio by a factor less than 5/4. (We can rescale weights so that $\beta = kn^2$ for some $k > 1$; this would make this factor very close to 1.) Now we may assume that $n^2 \leq \beta^* 2n^2$.

Let $g \approx 9cn \log n$.

We translate our instance of weighted MIN-BISECTION with n nodes and integer weights into an instance of unweighted MIN-BISECTION with gn nodes. We replace each node u with a clique of g nodes, say G_u. The edges between G_u and G_v depend on the weight of the edge $\{u, v\}$, $w(u, v)$. If $w(u, v) \geq 2n^2$, we say that the edge $\{u, v\}$ is *heavy*, in this case the connection between G_u and G_v will be a full bipartite graph. Otherwise, for some integers $a \leq 2n^2/g, b < g$ we will have $w(u, v) = ag - b$. We create a disjoint matchings between G_u and G_v, and from the last matching we remove b edges.

Suppose that $A \subseteq V$ is an optimum bisection of the original instance, *i.e.* $|A| = n/2$ and $cut(A) = \beta^*$. One can see that $Cut(A)$ does not contain any heavy edges, as the cost of a heave edge is larger then $2n^2$. We can transform A into $G_A = \bigcup_{u \in A} G_u$, it is easy to see that the values of corresponding cuts are equal, *i.e.* $cut(G_A) = cut(A)$.

Conversely, suppose that we have a bisection B for the new instance that is found by an approximation algorithm. Because we assume that this algorithm is at least as good as the one of Feige and Krauthgammer [FK00], we know that it produces a bisection with cut not larger then

$$\begin{aligned}(c\log(gn))^2 \times 2n^2 &< \text{(because } g < n^2\text{)}\\ (3c\log n)^2 \times 2n^2 &= \frac{2}{9}g^2.\end{aligned}$$

Assuming that $cut(B) \le \frac{2}{9}g^2$ we will find a solution A for the original instance such that $cut(A) \le cut(B)$. It suffices to show that we can modify B to a bisection B' such $cut(B') \le cut(B)$ and for every group G_u, either $G_u \subseteq B'$ or $G_u \subseteq \overline{B}'$, where $\overline{B}'$ is the complement of B'.

To define B', for every $u \in V$ we compare the sizes of $|G_u \cap B|$ and $|G_u \cap \overline{B}|$, the larger of these sets is called a *majority* of G_u, and the smaller its *minority*. Every element of a minority is connected to at least $g/2$ edges that go to the respective majority, and thus belong to $Cut(B)$. Because $cut(B) < \frac{2}{9}g^2$, all minorities have together less than $\frac{4}{9}g$ nodes. We define B' as the union of G_u's such that the majority of G_u is in B. We can show that B' is a bisection and $cut(B') \le cut(B)$.

B' is a bisection because $ng/2 - \frac{4}{9}g < |B'| < ng/2 + \frac{4}{9}g$ and $|B'|$ is a multiple of g, thus $|B'| = ng/2$.

It remains to show that $cut(B') \le cut(B)$.

Because the joint size of the minorities in B is equal to the joint size of minorities in the complement of B, no minority is larger than $\frac{2}{9}g$.

For some $G_u \subseteq B'$ let k be the number of complete matchings that connect G_u with groups that are contained in $\overline{B}'$. In such a matching, edges from majority to majority belong to $Cut(B)$, so the matching contains at least $\frac{5}{9}g$ edges of $Cut(B)$. Therefore $\frac{5}{9}gk < \frac{2}{9}g^2$, and thus $k \le \frac{2}{5}g$.

We change B into B' by moving minority nodes to the respective majorities. Consider moving of the minority node $x \in G_u \subseteq B'$; x is incident to at least $\frac{7}{9}g$ edges with the other end in the majority of G_u, these edges belong to $Cut(B) - Cut(B')$. For every edge in $Cut(B')$ that is incident to x we have a matching that connect G_u with a group contained in $\overline{B}'$, there are at most $\frac{2}{5}g + n/2 < \frac{1}{2}g$ such matchings. Consequently, moving x to the majority decreased the cut.

□

Theorem 1, combined with an approximation algorithm of [FK00] for minimum bisection on graphs entails

Theorem 2 *For every integer k, there exists a polynomial time approximation algorithm for MIN-Hk-BISECTION with approximation ratio $k \cdot c(\log n)^2$ for c a constant independent on k.*

Theorem 1 entails also the following MIN-BISECTION approximation hardness result relative to the hardness of MIN-Hk-BISECTION problems.

Theorem 3 *If MIN-Hk-BISECTION, for arbitrary $k \geq 3$, is hard to approximate to within an approximation ratio $r(n) > \frac{k}{3}$, then MIN-BISECTION is hard to approximate to within a ratio $3r(\sqrt[3]{n})/k$.*

4 Approximation Hardness of Very Sparse Instances of MIN-Hk-BISECTION

We study now approximation hardness of very sparse and regular instances of MIN-Hk-BISECTION for $k \geq 3$. We describe a translation from MIN-BISECTION problem to 2-MIN-Hk-BISECTION, which we define as a restriction of MIN-Hk-BISECTION problem to the *2-regular instances* in which each vertex belongs to exactly two hyperedges. We refer to [BK01] for the corresponding hardness results for MIN-BISECTION as well as the constructions used in the proofs.

Theorem 4 *Assume $k \geq 3$. If there exists an $(r(n), t(n))$-approximation algorithm for 2-MIN-Hk-BISECTION, then there exists an $(r(n^3), t(n^3))$-approximation algorithm for MIN-BISECTION.*

Proof. Hypergraphs that are instances of 2-MIN-Hk-BISECTION are in some sense *dual* to k-regular graphs. Thus we can rephrase the 2-MIN-Hk-BISECTION as an equivalent problem for such k-regular graphs:

> k-DUAL-BISECTION: Given a k-regular graph with $4n$ nodes and $2kn$ edges, color kn edges white and kn edges black, while minimizing the number of *mixed* nodes *i.e.* nodes adjacent to edges of different colors.

We present here only the proof for $k = 3$, since it is easy to generalize it for arbitrary k. We will reduce graph minimum bisection to 3-DUAL-BISECTION, so that a graph G with $2n$ nodes is translated into a 3-regular graph G' with $4n(3n-1)2n$ nodes.

We construct an instance G' of 3-DUAL BISECTION as follows. For each node u of G we create a subset of edges E_u of G': it has $2n-1$ rows numbered 1 to $2n-1$, each row consists of $6n$ nodes and $6n$ edges connected into a ring, and nodes of row i alternate between those connected to row $i-1$ and those connected to row $i+1$.

In rows 1 and $2n-1$ we have $6n$ nodes not connected to other rows, instead they are incident to $6n$ new distinct edges of E_u that are viewed as as $2n$ *triples*. To each neighbor of u, say v, we assign endpoints of one of these triples and we call them a_u^v, b_u^v, c_u^v. For a triple that is not assigned to a neighbor we create a new node connected to the elements of this triple.

If $\{u,v\}$ is an edge of G, then we create a pair of nodes $P_{\{u,v\}} = \{d,e\}$, we connect d to a_u^v, b_v^u and c_u^v and e to a_v^u, b_u^v and c_v^u as shown in Fig. refgadget1.

One can see that E_u contains $2n-1$ rings of $6n$ edges each, $3n$ edges between each pair of rings and $6n$ edges that are grouped in triples, so $|E_u| = (2n-1)\times 6n + (2n-2)\times 3n + 6n = 6n(3n-1)$ edges, thus we created $4n(3n-1)$ nodes per node in G.

Now, given an edge coloring of G', we *normalize* it so that each E_u is colored with one color. Such a coloring is equivalent to a bisection of nodes in G; an edge $\{u,v\}$ belongs to this bisection if and only if both elements of $P_{\{u,v\}}$ are mixed; consequently the ratio between the number of mixed nodes in a *normalized* coloring and the number of edges in the cut of the equivalent bisection is always 2. To show that finding a good coloring leads to an equally good bisection we need to show that we can normalize without increasing the number of mixed nodes.

Consider E_u such that at least n rows consist of edges of two colors. Such rows contain at least two mixed nodes, so the rings of E_u contain at least $2n$ mixed nodes. We move such E_u into the *reserve*; once we are done with re-coloring of other edge sets we set the color of E_u in such a way as to create the bisection; as a result it may happen that nodes of up to n pairs of the form $P_{\{u,v\}}$ will become mixed (only n because we know that v belongs to the other part in the bisection). This is amortized by $2n$ or more nodes from the rows of E_u that ceased to be mixed.

Similarly, suppose that E_u contains one purely white ring and one purely black ring. There exist $3n$ node disjoint paths connecting these two rings,

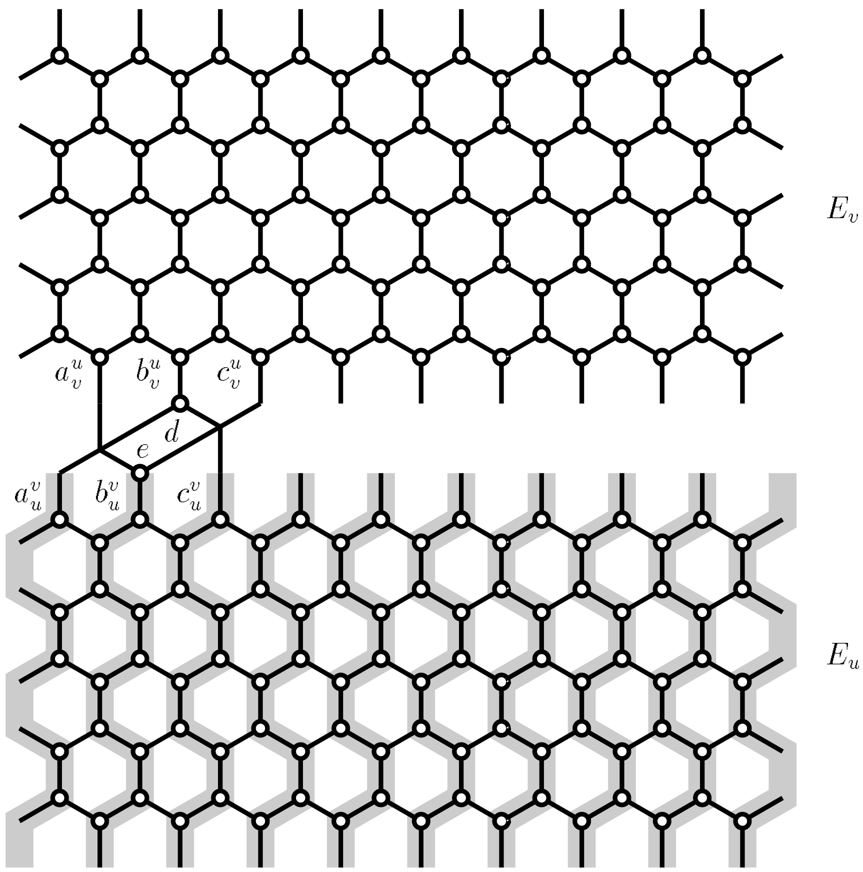

Figure 1: Edge sets E_u and E_v and the translation of edge $\{u, v\}$. The edge sets are completely depicted if $n = 3$ and we identify the leftmost edges (without left endpoints) with the rightmost edges (without right endpoints).

and each path must contain a mixed node. Again, we can move E_u into the reserve.

It remains to consider E_u such that at least n rows are completely white (black). Then we convert all edges of E_u to white (black).

The nodes of E_u can be partitioned into $3n$ disjoint connected paths, called columns that cover all the nodes in the rings, such paths are shown in gray in Fig. refgadget1. A column has exactly 2 nodes in each ring, and thus $4n - 3$ edges. We extend this path with edges adjacent to the first and the last node and which belong to the triples). We insert edges that connect two column to the column on the left, now each column has $6n - 2$ edges. We remove from each column edges that belong to purely white rings, this splits a column into connected components which we call sub-columns. Because a sub-column may have edges in at most $n-1$ rings, it may contain at most $3(n-1)+1=3n-2$ edges. We consider sub-columns in which at least one edge was black before the conversion to white. In each such sub-column at least one node ceases to be mixed after the conversion.

If a sub-column is not adjacent to a pair of the form $P_{\{u,v\}}$ we eliminate one mixed node after converting at most $3n-2$ edges.

Now consider a pair of the form $P_{\{u,v\}}$; in E_u it is adjacent to three sub-columns. If in each of these sub-columns we converted an edge, we eliminate at least 3 mixed nodes, but now the nodes of $P_{\{u,v\}}$ may become mixed, so the net gain is the elimination of at least 1 mixed node, while we converted at most $3(3n-2)$ edges. Suppose that we have converted an edge in two of these three columns. Then one of these columns was pure white, and the node of $P_{\{u,v\}}$ adjacent to this column was mixed or pure white. Thus we eliminate 2 mixed nodes in two columns and we get at most 1 new mixed node in $P_{\{u,v\}}$, for the net gain of 1 node, while we converted at most $2(3n-2)$ edges. It remains to consider the case when we converted an edge in only one of the 3 columns.

If this column contains a_u^v or c_u^v, then both nodes of $P_{\{u,v\}}$ are mixed or white, the conversion does not add any new mixed nodes and eliminates one. In the remaining case, columns of a_u^v and c_u^v are purely white. Suppose that the 4-edge path from a_u^v to c_u^v is not purely white, then our conversion removes two mixed nodes from this path while adding at most one new mixed node, e. In the remaining case we convert only edge $\{b_u^v, e\}$ without changing the number of mixed nodes.

Summarizing, either we gain at least 1 node while converting at most $3n(3n-2)$ edges, or we have a *special* case in which we have no loss or gain and we convert at most one edge.

Suppose that not counting the special cases we have converted K edges. To have a bisection, we may need to re-convert up to K edges — if there exists the reserve, or if we have converted both white and black edges then we need to re-convert fewer edges. Thus it suffices to re-convert at most $\lfloor \frac{K}{6n(3n-1)} \rfloor$ sets of the form E_u, which may create up to $2n\lfloor \frac{K}{6n(3n-1)} \rfloor \leq \lfloor \frac{K}{3(3n-1)} \rfloor$ new mixed nodes. But when we converted K edges we eliminated at least $\frac{K}{3n(3n-2)}$ mixed nodes, so there is no net loss. At this point we have a balance of the bisection except for the impact of special cases, However, there can be at most $2n(2n-1)/2 = n(2n-1)$ special cases, so we need to re-convert at most $n(2n-1)$ edges. This would be less than the size of a single E_u, which is not possible, therefore we already have achieved the balance of the bisection.

To adapt the above proof to k-DUAL-BISECTION, we would have to construct a k-regular graph rather than 3-regular one. We can do it by adding more edges inside node gadgets. The new edges will make it even less profitable to bisect the edge colorings that split node gadgets.

Thus, we get under assumption that there exists an $(r(n), t(n))$-approximation algorithm for 2-MIN-Hk-BISECTION for every integer $k \geq 3$, the existence of an $(r(n^3), t(n^3))$-approximation algorithm for MIN-BISECTION for arbitrary graphs.

□

We are going to prove now, using Theorem 4, approximation lower bounds for 2-MIN-Hk-BISECTION problems under average complexity assumption of [F02].

We refer to [F02] for the background on the average case complexity and its connection to the approximation complexity.

We will say that the optimization problem P is *R3SAT-hard* to approximate to within approximation ratio ρ, if the existence of a polynomial time algorithm for P within an approximation ratio ρ contradicts Hypothesis 2 of [F02].

Theorem 5 ([F02]). *MIN-BISECTION is R3SAT-hard to approximate to within a ratio below* $\frac{4}{3}$.

We have the following result on sparse hypergraphs:

Theorem 6 *For every integer* $k \geq 3$, *2-MIN-Hk-BISECTION is R3SAT-hard to approximate to within a ratio below* $\frac{4}{3}$.

Proof. By combining Theorems 4 and 5.

□

5 Further Research

An interesting open question remains whether the dependence on the dimension k in Theorem 1 can be somehow reduced. This will require a new *sparser* reduction in a dimension of a hypergraph than the one used in Theorem 1.

Holmerin and Khot in [H02], [HK03] prove approximation hardness of 4- and 3-dimensional equational extension of MIN-BISECTION (this extension however does not define valid hypergraph bisections). A very interesting question is whether there is any meaningful connection between the hardness of the 3-dimensional or higher dimensional equational extension and the problem of MIN-Hk-BISECTION. Such a connection might ultimately shed some light on approximation hardness of the minimum graph bisection.

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