On Approximability of Minimum Bisection and Related Partition Problems*

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Abstract. We survey some recent results on the complexity of computing approximate solutions for instances of the *Minimum Bisection* problem and formulate some intriguing and still open questions about the approximability status of that problem. Some connections to other optimization problems are also indicated.

1 Introduction

The problem of approximating the minimum bisection of a graph, i.e., the problem of partitioning a given graph into two equal halfs so as to minimize the number of edges with exactly one end in each half, belongs to the most intriguing problems currently in the area of combinatorial optimization and the approximation algorithms. The reason being that we are not able to cope at the moment with the global conditions imposed on the vertices of a graph like the condition that the two parts of a partition are of equal size. The MIN-BISECTION problem arises profoundly in several contexts, either explicitly or implicitly, which range from problems of statistical physics and combinatorial optimization to computational geometry and various clustering problems, (cf., e.g., [MPV87], [JS93], [H97]). We refer also to [PY91] and [AL97] for the background on approximation algorithms and approximation hardness of optimization problems.

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2 Instances of MIN-BISECTION Problem

We are going to define the instances of the MIN-BISECTION problem studied in our paper.

- *MIN-BISECTION:* Given an undirected graph, partition the vertices into two equal halves so as to minimize the number of edges with exactly one endpoint in each half.
- *Paired MIN-BISECTION:* Given an undirected graph, and a set of pairs of its vertices, partition the vertices into two equal halves so as to split each given pair of vertices and to minimize the number of edges with exactly one endpoint in each half.
- *Weighted MIN-BISECTION:* Given a *weighted* undirected graph, partition the vertices into two equal halves so as to minimize the sum of weights of the edges with exactly one endpoint in each half.

We refer to a graph $G = (V, E)$ as a *dense* graph if its *minimal* degree is $\Theta(n)$. We call a graph G *planar*, if G can be embedded into a plane graph. A *weighted* complete graph G is called *metric*, if G can be embedded into a finite metric space.

- *Dense MIN-BISECTION* is the MIN-BISECTION problem restricted to the dense graphs.
- *Dense Paired MIN-BISECTION* is the Paired MIN-BISECTION problem restricted to the dense graphs.
- *Planar MIN-BISECTION* is the MIN-BISECTION problem restricted to the planar graphs.
- *Metric MIN-BISECTION* is the Weighted MIN BISECTION problem restricted to the metric graphs.

We define, in a similar way, the dual MAX-BISECTION problems for the general, dense, planar and metric graphs, respectively.

It is not difficult to see that the *dense* and *metric* instances of MIN-BISECTION and Paired MIN-BISECTION are both NP-hard in exact setting (cf. [AKK95], [BF99], [FK98b]). It was proven recently that the Planar MAX-BISECTION ([J00], see also [JKLS01]) is *NP-hard* in *exact* setting, however the status of the Planar MIN-BISECTION remains still an intriguing open problem.

We refer to [K01a] and [K01b] for a survey on approximability of dense and sparse instances of some other NP-hard combinatorial optimization problems.

3 Dense Instances of MIN-BISECTION, Paired MIN-BISECTION, and MIN-2SAT

We consider here also the following minimization problems.

- *MIN-2SAT:* Given a 2CNF formula, construct an assignment as to minimize the number of clauses satisfied.

We refer to the 2CNF formula to be *dense* if the number of occurrences of each variable is $\Theta(n)$.

- *Dense MIN-2SAT* is the MIN-2SAT problem restricted to the dense formulas.

It is known that the large fragments of Minimum Constraint Satisfaction (MIN-CSP) problems do not have polynomial time approximation schemes even if restricted to the dense instances (see [CT96], [KZ97], [BFK01]). It has turned out however, a bit surprisingly, that the dense instances of MIN-BISECTION do have a PTAS [AKK95].

Theorem 1. ([AKK95]) *There exists a PTAS for Dense MIN-BISECTION.*

The method used in [AKK95] depended on a new technique of approximating Smooth Polynomial Integer Programs for large values of objective functions, and a biased *radical* placement method for the small values. The variant of that technique was used in Bazgan and Fernandez de la Vega [BF99] to prove that dense instances of Paired MIN-BISECTION possess a PTAS.

Theorem 2. ([BF99]) *There exists a PTAS for Dense Paired MIN-BISECTION.*

The above result was used to derive the existence of a PTAS for dense instances of MIN-2SAT. It has turned however out that the proof in [BF99] contained an error. The corrected proof was established in Bazgan, Fernandez de la Vega and Karpinski [BFK01].

Theorem 3. ([BFK01]) *There exists a PTAS for Dense MIN-2SAT.*

We notice that both Paired MIN-BISECTION and MIN-2SAT are both provably MAX-SNP-hard (cf. [BF99], [KKM94]), and thus not having PTASs under usual complexity theoretic assumptions. Intriguingly, all attempts to establish a connection between the approximation hardness of Paired MIN-BISECTION and MIN-2SAT and the approximation hardness of MIN-BISECTION have failed utterly up to now. The approximation hardness status of MIN-BISECTION remains an outstanding open problem. At the moment we are not even able to exclude a possibility of existence of a PTAS for that problem.

Open Problem 1. *Is MIN-BISECTION NP-hard to approximate to within a constant factor?*

On the positive side, there was recent substantial improvement on approximation ratio for MIN-BISECTION, cf. Feige, Krautghamer and Nissim [FKN00], and Feige and Krautghamer [FK00].

Theorem 4. ([FK00]) *MIN-BISECTION can be approximated in polynomial time to within $O(log^2 n)$ factor.*

[FK00] gives also an improved approximation factor for planar instances of MIN-BISECTION.

Theorem 5. ([FK00]) *Planar MIN-BISECTION can be approximated in polynomial time to within $O(logn)$ factor.*

4 Planar Instances of MIN-BISECTION

There has been a very recent progress on the approximability status of planar MAX-BISECTION resulting in design of the first PTAS for that problem, and also in the first proof of its NP-hardness in exact setting [J00], [JKLS01].

The status of planar MAX-BISECTION was an open problem for a long time. An intriguing context for that problem is the fact that planar MAX-CUT can be computed exactly in polynomial time [H75]. An additional paradigm connected to it was based on analysis of cut polytops, and the fact that the value of the planar MAX-CUT semidefinite relaxation with *triangle constraints* is just equal to the value of the optimal cut (cf. [BM86] for the background). The corresponding problem for *bisectional* polytops however remains still open.

The *exact* computation status for planar MAX-BISECTION was resolved recently by Jerrum [J00] (cf. also [JKLS01]) in proving its NP-hardness. The technique of his proof is similar to the method used by Barahona [B82] for the planar *spin glass* problem within a *magnetic field*, and is based on the NP-hardness of the maximum independent set on 3-regular planar graphs.

Theorem 6. ([J00]) *Planar MAX-BISECTION is NP-hard in exact setting.*

Soon after Jansen, Karpinski, Lingas and Seidel [JKLS01] were able to design the first PTAS for planar MAX-BISECTION, and for some special cases of planar MIN-BISECTION. The method of solution depended on a new method of finding maximum partitions of bounded treewidth graphs, combined with the tree-type dynamic programming method of dividing planar graph into k-outerplanar graphs [B83].

Theorem 7. ([JKLS01]) *There exists a PTAS for Planar MAX-BISECTION.*

We notice that Theorem 6 and 7 do not entail readily any corresponding result for planar MIN-BISECTION. The reason being that the operation of *complementing* an instance of planar MAX-BISECTION does not result in a planar instance of MIN-BISECTION (alike some other situations).

However, the results of [JKLS01] entail also the following.

Theorem 8. ([JKLS01]) *There exists a PTAS for instances of the Planar MIN-BISECTION with a size of minimum bisection $\Omega(n \log \log n/\log n)$.*

The proof of Theorem 8 depends on the fact that the PTAS of Theorem 7 works for the partitionings of the treewidth up to $O(\log n)$. We observe, by the planar separator theorem [LT79] for bounded degree planar graphs, that the size of minimum bisection is $O(\sqrt{n})$. This fact yields also

Theorem 9. ([JKLS01]) *Given an instance G of Planar MIN-BISECTION of size n and maximum degree d, a minimum bisection of G of size $O(d\sqrt{n})$ can be computed in time $O(n \log n)$.*

The problem on whether Planar MIN-BISECTION admits PTAS, or perhaps even polynomial time exact algorithms, remains open.

Open Problem 2. *Is Planar MIN-BISECTION NP-hard in exact setting?*

Open Problem 3. *Does Planar MIN-BISECTION have a PTAS?*

We will study in the next sections the case of metric MIN-BISECTION, and connected problems of metric MIN-CLUSTERING.

5 Metric Instances of MIN-BISECTION and MIN-PARTITIONING

Metric (and more restrictively, geometric) instances of combinatorial optimization problems occur in a number of realistic scenarios and are strongly motivated by various applications (cf.[H97]). The instances of such problems are given by embeddings in finite metric spaces.

We consider first metric MAX-CUT, i.e. the problem of partitioning a given finite metric space (V, d) so as to maximize the sum of distances across the partition.

Fernandez de la Vega and Kenyon [FK98b] were the first to design a PTAS for the Metric MAX-CUT. Their method followed the earlier work of Fernandez de la Vega and Karpinski [FK98a] on existence of a PTAS on dense weighted instances of MAX-CUT.

Theorem 10. ([FK98b]) *There exists a PTAS for Metric MAX-CUT.*

In a very recent work, Fernandez de la Vega, Karpinski and Kenyon [FKK03] resolved finally the status of the Metric MIN-BISECTION problem by proving an existence of a PTAS for that problem. The method of the solution depends on a new kind of biased sampling and a new type of randomized rounding. The method of the solution could be also of independent interest.

Theorem 11. ([FKK03]) *There exists a PTAS for Metric MIN-BISECTION.*

The method of [FKK03] gave also rise to the consecutive solution for Metric MIN-kCLUSTERING problems, see the next Section.

We are going to extend now the notion of the metric minimum bisection to the notion of size constranted K-ary minimum partioning of a finite metric space, as well as metric MIN-k-CUT problems.

- *K-ary metric MIN-PARTITIONING:*
  Given a finite metric space (V,d) of size n, and a sequence of sizes $(n_1, n_2, \ldots, n_k)$ such that $\sum_{i=1}^{n} n_i = n$, construct a partition of V into K parts of sizes $(n_1, n_2, \ldots, n_k)$ so as to minimize the sum of distances between the different parts.

- *Metric MIN-k-CUT:*
  Given a finite metric space (V,d), construct a partitioning of V into k parts so as to minimize the sums of distances between different parts.

- *Metric MIN-MULTIWAY-k-CUT:*
  Given a finite metric space (V,d) and a set of k terminals $T \subseteq V$, construct a partition of V so as to disconnect every terminal from each other and to minimize the sums of distances between different parts.

The paper [FKK03] gives the first PTASs for the above problems.

Theorem 12. ([FKK03]) *There exists a PTAS for K-ary Metric MIN-PARTITIONING.*

Theorem 13. ([FKK03]) *There exist PTASs for Metric MIN-k-CUT and Metric MIN-MULTIWAY-k-CUT.*

6 Metric Instances of MIN-CLUSTERING Problems

We introduce now the class of important, in a sense, dual *metric clustering problems.*

- *Metric MIN-k-CLUSTERING:*
  Given a finite metric space (V, d), partition V into k sets $C_1, C_2, \ldots, C_k$ so as to minimize the sum $\sum_{i=1}^{k} \sum_{x,y \in C_i} d(x, y)$ (called the sum of intra-cluster distances).

The status of metric MIN-2-CLUSTERING problem was left open in [FK98b], and solved later by Indyk in [I99]. The main difficulty to be overcome in Indyk's solution was copying with the situations where the value of max-cut was much higher than the value of the 2-clustering.

Theorem 14. ([I99]) *There exists a PTAS for Metric MIN-2CLUSTERING.*

Finaly, Fernandez de la Vega, Karpinski, Kenyon and Rabani [FKKR02] proved the following general result on metric MIN-k-CLUSTERING.

Theorem 15. ([FKKR02]) *There exists a PTAS for Metric MIN-k-CLUSTERING for each k.*

7 Sparse Instances of MIN-BISECTION

We turn our attention to the dual class, i.e. to the class of *sparse* instances. For the representative of this class we choose the class of 3-regular graphs.

We introduce first some notation. We will call an approximation algorithm A for an optimization problem P, an $(r(n), t(n))$*-approximation algorithm*, if A approximates P within a factor $r(n)$, and running time of A is $O(t(n))$ for n the size of an instance.

The following result has being proven recently by Berman and Karpinski [BK01].

Theorem 16. ([BK01]) *Suppose there exists an* $(r(n), t(n))$*-approximation algorithm for 3-regular instances of MIN-BISECTION. Then there exists an* $(r(n^3), t(n^3))$*-approximation algorithm for MIN-BISECTION on general instances.*

The construction of [BK01] can be modified as to yield a similar result on 3-regular planar graphs. In such a modification we use a slightly larger piece of hexagonal mesh, and replace each edge between a pair of nodes with a pair of edges between meshes that replaced those nodes (cf. [BK01]).

Theorem 17. *Suppose there exists an $(r(n), t(n))$-approximation algorithm for 3-regular instances of Planar MIN-BISECTION. Then there exists an $(r(n^3), t(n^3))$ -approximation algorithm for Planar MIN-BISECTION.*

Theorem 16, and 17 give a relative hardness of 3-regular instances of MIN-BISECTION. The approximation lower bound status of MIN-BISECTION, and, in fact, the exact computation lower bound for Planar MIN-BISECTION, remain important and intriguing open problems.

It is also interesting to notice that the recent improvements in approximation ratios for 3-regular instances of MAX-BISECTION (cf. e.g. [FKL00], [KKL00]) were not paralleled by the analogous improvements on the 3-regular instances of MIN-BISECTION. Theorem 16, and 17 give good reasons for this development.

As for the general MIN-BISECTION problem, Feige [F02] was able to prove recently a relative hardness result on approximating it within some constant factor, connecting it to a hypothesis on the approximate hardness of random 3SAT formulas (cf. for details [F02]).

8 Summary of Known Approximation Results for MIN-BISECTION

We present here (Table 1) the best up to now approximation upper and lower bounds for the instances of MIN-BISECTION problem.

Instances	Approx. Upper	Approx. Lower
General	$O(\log^2 n)$	Not known
Dense	PTAS	–
Sparse	$O(\log^2 n)$	Equal to MIN-BISECTION
Planar	$O(\log n)$	Not known to be NP-Hard even in exact setting
Sparse Planar	$O(\log n)$	Equal to Planar MIN-BISECTION
Metric	PTAS	–

Table 1
Approximation Upper and Lower Bounds for MIN-BISECTION

9 Further Research

The most challenging and intriguing open problem remains the status of the MIN-BISECTION on general graphs. The known so far PCP-techniques do not seem to yield any approximation lower bounds for that problem. The same holds for the known techniques to approximate MIN-BISECTION. They do not seem to allow us to break the approximation factor at any level below $O(logn)$, and this even for 3-regular planar graphs. It seems that the improved upper and lower approximation bounds for MIN-BISECTION will require essentially new techniques. This holds not only for the general MIN-BISECTION, but also for the very restricted Planar MIN-BISECTION instances, for which we are not able to prove at the moment even NP-hardness for the exact computation, nor are we able to give any better than $O(logn)$ approximation factors.

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