A Polynomial Time Approximation Scheme for Subdense MAX-CUT

W. Fernandez de la Vega[*]   Marek Karpinski[†]

Abstract

We prove that the subdense instances of MAX-CUT of average degree $\Omega(n/logn)$ posses a *polynomial time approximation scheme* (PTAS). We extend this result also to show that the instances of general 2-ary *maximum constraint satisfaction problems* (MAX-2CSP) of the same average density have PTASs. Our results display for the first time an existence of PTASs for these subdense classes.

1

1 Introduction

Significant recent results concerning existence of polynomial time approximation schemes (PTASs) for *dense* instances of several $\mathcal{NP}$-hard problems such as MAX-CUT, MAX-k-SAT, BISECTION, DENSE-k-SUBGRAPH, and dense MAX-SNP problems have been obtained in Arora, Karger and Karpinski [AKK95], Fernandez de la Vega [F96], Arora, Frieze and Kaplan [AFK96], Frieze and Kannan [FK97]. Still more recently, the approximability of dense instances of $\mathcal{NP}$-hard problems has been investigated from the point of view of the query complexity. Goldreich, Goldwasser and Ron [GGR96] show that a constant size sample is sufficient to test whether a graph has a cut of a certain size. Frieze and Kannan [FK97] obtained faster approximations for all dense MAX-CSP problems. Alon, Fernandez de la Vega, Kannan and Karpinski [AFKK01] succeeded further in improving efficiency and sample complexity of the underlying approximations of dense MAX-CSP classes. Recall that a PTAS for a given optimization problem is a family (A_ϵ) of algorithms indexed by a parameter $\epsilon \in (0, \infty)$ where each algorithm runs in polynomial time and, for each ϵ, the algorithm A_ϵ has approximation ratio $1 - \epsilon$ (or $1 + \epsilon$ for a minimization problem). In most cases, the instances are graphs, and a dense graph is defined as a graph with $\Theta(n^2)$ edges where n is the number of vertices. (In some cases, the algorithms apply only to graphs with minimum degree $\Theta(n)$.) Some of the problems considered in the papers mentioned above, such as MAX-CUT, are MAX-SNP-hard, and thus, if $\mathcal{P} \neq \mathcal{NP}$, have no PTASs when the set of instances is not restricted.

In this paper, we adress the question of whether the density condition can be relaxed. This possibility was anticipated in [AKK95]. The next two theorems give a partial answer to this query.
Recall that the density $d(G)$ of a graph G is defined by

$$d(G) = \binom{n}{2}^{-1} |E(G)|,$$

and an *average degree* is defined as $d(G)n$.

Theorem 1. MAX-CUT problem *does have a* PTAS *on the set of graphs G with density* $\Omega(1/\log n)$.

We generalize our result to subdense classes of general MAX-2CSP problems (cf. for definitions [KSW97]).

A *density* $d(\mathcal{C})$ of an instance of an MAX-2CSP problem is defined by $d(\mathcal{C}) = \binom{n}{2}^{-1} N$ for N a number of constraints of $\mathcal{C}$.

Theorem 2. MAX-2CSP *has a* PTAS *on the set of instances of density* $\Omega(1/\log n)$.

2 Proof of Theorem 1

2.1 Results on Representativity.

Assume that we have an instance $G = (V, E)$ of MAX-CUT with $|V| = n$ vertices and $dn^2/2$ edges where $d = \Omega(1/\log n)$. The weight of a set of vertices $U \subseteq V$ is defined as the sum of the degrees of the vertices in U. In particular, we define $W = dn^2$ as the weight of V. For consistency of notation, we write w_v for the degree of v, whereas $\Gamma(v)$ will denote the set of neighbors of v in G. We follow a variant of a concept introduced in [GGR96] (see also [FKK02] and [F96]) of, so called, *set representativity*. To suit our purpose, we formulate it as follows.

Definition 1 *For any subset $T \subseteq V$ and $v \in V$, let*

$$\gamma(v, T) = |\Gamma(v) \cap T|.$$

*Consider a partition $P = (L, R)$ of V. For $T \subseteq V$, define $T_L = T \cap L, T_R = T \cap R$. T is called (δ, ϵ)-**representative** with respect to P if for every vertex v except perhaps for a subset of **exceptional** vertices of weight at most δW, we have*

$$\left| \frac{n\gamma(v, T_L)}{|T|} - \gamma(v, L) \right| \leq \epsilon w_v,$$

and

$$\left| \frac{n\gamma(v, T_R)}{|T|} - \gamma(v, T) \right| \leq \epsilon w_v.$$

*A vertex which is not exceptional is called **normal**.*

We will need $(\epsilon^2, \epsilon/10)$-representativity. Let us show that we can achieve this with a suitable $t = |T|$.

Lemma 1 *Let t be any fixed integer ≥ 1. Let T be a random sample (possibly with ties) obtained by picking independently with replacement t points $u_i \in V$ with the uniform distribution. Let $v \in V$ be any fixed vertex and let $T_L, T_R, \gamma(v, T_L)$ and $\gamma(v, T_R)$ be defined as in definition 1. Then we have that*

$$\Pr\left(\left| \frac{n\gamma(v, T_L)}{t} - \gamma(v, L) \right| \geq \epsilon w_v \right) \leq \exp(-\frac{\epsilon^2 t w_x}{2|V|}) \tag{1}$$

and,

$$\Pr\left(\left| \frac{n\gamma(v, T_R)}{t} - \gamma(v, R) \right| \geq \epsilon w_v \right) \leq \exp(-\frac{\epsilon^2 t w_x}{2|V|}) \tag{2}$$

Proof: We have that $\gamma(v, T_L)$ is Binomial with parameters t_L and $p = \frac{\gamma_L}{|V|}$ with $\gamma_L = |\Gamma(v) \cap L|$. Thus, by Hoeffding- Chernoff,

$$
\begin{aligned}
\Pr\left(\left|\frac{n\gamma(v, T_L)}{t} - \gamma(v, L)\right| \geq \epsilon w_v\right) &= \Pr\left(\left|\gamma(v, T_L) - \frac{t\gamma(v, L)}{n}\right| \geq \frac{t\epsilon w_v}{n}\right) \\
&\leq 2\exp\left(-\frac{\epsilon^2 t w_v^2}{2n\gamma(v, T_L)}\right) \\
&\leq 2\exp\left(-\frac{\epsilon^2 t w_v}{2n}\right) \\
&\leq 2\exp\left(-\frac{\epsilon^4 t d}{2n}\right)
\end{aligned}
$$

if $w_v \geq \frac{\epsilon^2 dn}{2}$ and

$$
\Pr\left(\left|\frac{n\gamma(v, T_L)}{t} - \gamma(v, L)\right| \geq \epsilon w_v\right) \leq \epsilon^3/20
$$

if $w_v \geq \frac{\epsilon^2 dn}{2}$ and $t \geq 10\log(1/\epsilon)/(d\epsilon^4)$ and ϵ is sufficiently small. Note that this choice of t has logarithmic size for $d = \Omega(1/\log n)$ and we can thus afford as we shall do to perform exhaustive search on the bi-partitions of T ∎

We can now prove that $(\epsilon^2 - \epsilon/10)$-representativity holds.
Let T be a random sample of V with size $|T| = t$, defined as just above and let (L, R) be an arbitrary bipartition of V. As we have just proved, the inequalities 1 and 2 hold for any fixed vertex of weight at least $\frac{\epsilon^2 dn}{2}$ with probability at least $1 - \epsilon^3/20$ implying that the total weight of the exceptional vertices has expectation at most $\frac{W\epsilon^3}{20}$. By Markov inequality, the weight of these vertices will not exceed $\frac{W\epsilon^2}{2}$ with probability $1 - \epsilon/10$. Adding the weight of the small vertices gives the claimed total $W\epsilon^2$. This proves:

Lemma 2 *Fix $t = 10\log(1/\epsilon)/(d\epsilon^4)$. Then, with probability at least $1 - \epsilon/10$, T is $(\epsilon^2, \epsilon/10)$-representative with respect to (L, R).*

Proof: See above.

∎

Lemma 3 *Let $V_1, V_2, ...V_\ell$ be a random partition of V in to sets of cardinality $n\epsilon$. Then with probability at least $1 - \epsilon^2$ we have:$\sum_{v \in V_j} w_v \leq 2\epsilon W$. With probability $\geq 9/10$ we have $\sum_j \sum_{u,v \in V_j} \gamma(u, v) \leq 11\epsilon W$.*

Proof: The proof is straightforward by using Markov inequality.

∎

2.2   The Algorithm

The algorithm takes as input a graph $G(V, E)$ on n vertices with density $d = \Omega(1/\log n)$. It makes a series of guesses and returns with probability at least $4/5$ when all these guesses are correct a cut of G whose value is within $(1 + O(\epsilon))$ of the optimum. We let $t = 2d \log(3/\epsilon)/\epsilon^4$.

1. Compute vertex weights $w_v = \mathrm{degree}(v)$ and total weight $W = \sum_v w_v = 2|E(G)|$.

2. Let $\ell = 1/\epsilon$ and define a partition $V_1, V_2, ..., V_\ell$ of V by placing each vertex in a randomly chosen V_j.

3. Let $P_o = (L, R)$ be an optimum cut of G. Let (L_j, R_j) be the partition of V_j induced by P_o. In the next phase the algorithm will construct inductively a sequence of "hybrid" partitions $P_0, P_1, ..., P_j, ..., P_\ell$ where the first hybrid is P_0, the last partition P_ℓ is the output, and such that, for each fixed j, P_j coincides with P_0 on each of the sets $V_{j+1}, V_{j+2}, ..., V_\ell$:

4. For each $j = 1, 2, \ldots, \ell$, do the following :

   (a) Let T_{j-1} denote a random multiset of V obtained by picking t times a vertex v of V according to the uniform probability distribution on V.

   (b) By exhaustive search, guess the partition (T'_{j-1}, T''_{j-1}) induced on T_{j-1} by $(A_1, B_1), \ldots, (A_{j-1}, B_{j-1}), (L_j, R_j), (L_{j+1}, R_{j+1}j) \ldots, (L_\ell, R_\ell)$. That is, classify the vertices of T_{j-1} which are in $V_1, V_2, \ldots, V_{j-1}$ according to the partition being built by the algorithm, and classify the remaining vertices of T_{j-1} according to the optimal partition .

   (c) For $v \in V_j$, let

   $$\hat{b}(v) = |\Gamma(v) \cap T"_{j-1}| - |\Gamma(v) \cap T'_{j-1}|$$

   (d) Construct a partition (A_j, B_j) of V_j by placing the $|V_j|/2$ vertices of V_j with non-negative values of $\hat{b}(v)$ in A_j and the others in B_j.

   Let $A = \cup_j A_j$ and $B = \cup_j B_j$.

5. Output the best of the cuts (A, B) thus constructed.

2.3   The Analysis

Recall that for each $j \in \{0, ..., \ell\}$ P_j is the partition which agrees with the partitions $(A_1, B_1), ..., (A_j, B_j)$ constructed by the algorithm in $V_1, ..., V_j$, and which agrees with the optimal partition (L, R) in $V_{j+1}, ..., V_\ell$. We let EX_j denote the set of exceptional vertices occuring in the jth phase.

Definition 2 *Consider a partition $P = (L, R)$ of V. The unbalance of a vertex $v \in V$ with respect to P is the quantity*

$$\widehat{ub}(v) = |\Gamma(u) \cap R| - |\Gamma(u) \cap L|$$

Lemma 4 *If T_{j-1} is representative with respect to P_{j-1}, then we have that*

$$\text{COST}(P_{j-1}) - \text{COST}(P_j) \le 2\epsilon \sum_{v \in V_J} w_v + \sum_{v \in EX_j} w_v + W_j$$

where W_j denotes the number of edges inside V_j.

Proof: Let U_j denote the set of vertices which are placed differently in P_{j-1} and P_j. Clearly $U_j \subseteq V_j$. Let $u \in U_j$ and let $P_{j-1}(u)$ be the partition obtained from P_{j-1} by changing the side of U. We have then that:

$$\text{COST}(P_{j-1}) - \text{COST}(P_j) \le \sum_{u \in U_J} (\text{COST}(P_{j-1}) - \text{COST}(P_j(u)) + W_j. + \sum_{v \in EX_j} w_v + W_j.$$

Assume that $\widehat{ub}(u)$ is non-negative which means that u is on the left-side (L) of P_{j-1}. We have

$$(\text{COST}(P_{j-1}) - \text{COST}(P_j(u)) = |\Gamma(u) \cap R| - |\Gamma(u) \cap L|$$

Assume that u is normal. [Otherwise the contribution of u to the loss is bounded above by its weight and counted separetely.]. Then, the first term in the right-side of the above is approximated within ϵw_u by the quantity $\frac{|V|}{|T|}|\Gamma(u) \cap T_R|$ and the second term is approximated by the quantity $\frac{|V|}{|T|}|\Gamma(u) \cap T_R|$. Thus we get that

$$\text{COST}(P_{j-1}) - \text{COST}(P_j(u) \le \widehat{ub}(u)(u) + 2\epsilon w_u,$$

Summing over $u \in V_j$ gives us then the lemma.

∎

Lemma 5 *With probability at least 4/5, we have that*

$$\text{COST}(P_0) - \text{COST}(P_\ell \le 14\epsilon W$$

Proof: Observe that Lemma 2 holds simultaneously for all j with probability at least 9/10 and also that our bound $11\epsilon W$ for the total number of edges inside the V_j holds with probability 9/10. Summing the bounds given by the preceding lemma for each j gives the claimed result, with probability at least 4/5.

∎

6

3 Proof of Theorem 2

A PTAS for MAX-2CSP with density $\Omega(1/\log n)$ can be given along the same lines as our PTAS for subdense MAX-CUT. The only really new feature is an adequate version of representativity. We restrict ourselves to formulate this new version and show that it holds with suitable values of the parameters, again with the logarithmic sample size.

Let $\mathcal{C}$ be an instance of MAX-2CSP on a set V of n boolean variables. As usual, we define the density of $\mathcal{C}$ by

$$d = \frac{|\mathcal{C}|}{n^2}$$

We denote by $N = |\mathcal{C}|$ the number of constraints in the instance. Fix an assignment a and let T denote a fixed subset of V. For each variable x let $n^1(x)$ (resp. $n^0(x)$) denote the number of constraints containing x which are true when x is set to true [resp. to false] and the other variables are set according to a. Let $n(x)$ be the total number of occurrences of x. We also refer to $n(x)$ as the weight of x. Let $n^1(x, T)$ [resp. $n^0(x, T)$] denote the number of constraints containing x and another variable in T which are true when x is set to true [resp. to false] and the other variables are set according to a.

Definition 3 (Representativity for MAX-2CSP) *The set of variables T is said to be $(\delta - \epsilon)$-representative with respect to the assignement a if for every variable v except perhaps for a subset of* **exceptional** *variables of weight at most δW, we have*

$$\left| \frac{n}{|T|} n^1(x, T) - n^1(x) \right| \leq \epsilon n(x)$$

and

$$\left| \frac{n}{|T|} n^0(x, T) - n^0(x) \right| \leq \epsilon n(x).$$

The following lemma is proved as its MAX-CUT counterpart.

Lemma 6 *Let t be any fixed integer ≥ 1. Let T be a random, uniformly picked sample of the variables with size $t = |T|$. Let $x \in V$ be any fixed variable and let $n(x), n^1(x), n^1(x, T), n^0(x), n^0(x, T)$ be defined as above. Then we have that, for $i=0,1$,*

$$\Pr \left(\left| \frac{n}{|T|} n^i(x, T) - n^i(x) \right| \leq \epsilon n(x) \right) \geq 1 - 2 \exp \left(-\frac{\epsilon^2 t n(x)^2}{32 n n^i(x)} \right) \tag{3}$$

[The 32 in the right-hand side of 4 comes from the fact that there are 16 distinct boolean functions of 2 variables.]

Assuming that $n(x) \geq \epsilon^2 dn$, we get that, for $i = 0, 1$,

$$\Pr\left(\left|\frac{n}{|T|} n^i(x, T) - n^i(x)\right| \leq \epsilon n(x)\right) \geq 1 - 2\exp\left(-\frac{\epsilon^4 td}{32}\right)$$

Now assume furthermore $t = 150 \log(1/\epsilon)/(\epsilon^4 d)$. This gives

$$\Pr\left(\left|\frac{n}{|T|} n^i(x, T) - n^i(x)\right| \leq \epsilon n(x)\right) \geq 1 - \epsilon^3/20$$

for sufficiently small ϵ, again for $i = 0, 1$. Since the total weight of the variables x with $n(x) \leq \epsilon^2 dn$ clearly does not exceed $\epsilon^2 dn^2 \leq \epsilon^2 N$, we have, reasoning as in the previous section, that for our choice of t, T is $(\epsilon^2, \epsilon/10)$-representative.

Once we know that the above representativity property holds, the design of a PTAS for subdense MAX-2CSP is similar to the design of the PTAS for MAX-CUT of the preceding section.

4 Open problems

This work raises the following questions. What about subdense MAX-rCSP problems for arbitrary r? Our method of proving of Theorem 2 gives an apparently much weaker result in the case of $r \geq 3$. Can our method be extended to some other even more relaxed density classes of MAX-CUT and MAX-2CSP?

Acknowledgments. We thank Ravi Kannan and Claire Kenyon for stimulating discussions and remarks.

References

[AFK96] S. Arora, A.M. Frieze and H. Kaplan, A New Rounding Procedure for the Assignment Problem with Applications to Dense Graph Arrangements, *Proc. 37th IEEE FOCS (1996) pp.21-30*

[AFKK01] N. Alon. W. Fernandez de la Vega, R. Kannan, and M. Karpinski, *Random Sampling and MAX-CSP Problems*, ECCC Research Report TR01-100 (2001), also in Proc. 34th ACM STOC (2002), pp. 534-543.

[AKK95] S. Arora, D. Karger, and M. Karpinski, *Polynomial Time Approximation Schemes for Dense Instances of NP-Hard Problems*, In Proc. 27th ACM STOC (1995) 284-293, also in Journal of Computer and Systems Sciences 58 (1999) 193-210.

[F96] W. Fernandez de la Vega, MAX-CUT has a Randomized Approximation Scheme in Dense Graphs, *Random Structures and Algorithms*, Vol. 8. No.3 (1996) pp.187-198

[FK96] A.M. Frieze and R. Kannan, The Regularity Lemma and Approximation Schemes for Dense Problems, *Proc. 37th FOCS (1996) 12-20.*

[FK97] A.M. Frieze and R. Kannan, Quick Approximation to Matrices and Applications, Combinatorica 19 (2) (1999) pp 175-120.

[FKK02] W. Fernandez de la Vega, M. Karpinski, and C. Kenyon, A Polynomial Time Approximation Scheme for Metric MIN-BISECTION, ECCC, Technical Report TR02-025, 2002.

[GGR96] O. Goldreich, S. Goldwasser and D. Ron, Property Testing and its Connection to Learning and Approximation, *Proc. 37th FOCS (1996) 339-348*, the full paper appeared in J. ACM 45 (1998), 653-750.

[KSW97] S. Khanna, R. Sudan and D. Williamson, *A complete Classification of the Approximability of Maximazation Problems Derived from Boolean Constraint Satisfaction*, Proc. 29th ACM STOC (1997), pp. 11-20.

[PY91] C. H. Papadimitriou and M. Yannakakis. *Optimization, Approximation and Complexity Classes*, Journal of Computer and System Sciences, 43 (1991) 425-440.