A Polynomial Time Approximation Scheme for Metric MIN-BISECTION

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Abstract

We design a *polynomial time approximation scheme* (PTAS) for the problem of Metric MIN-BISECTION of dividing a given *finite metric* space into two halves so as to minimize the sum of distances across that partition. The method of solution depends on a new metric placement partitioning method which could be also of independent interest.

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1 Introduction

The MIN-BISECTION problem of dividing a given graph into two equal halves so as to *minimize* the number of edges or the sum of their weights across the partition belongs to the most intriguing problems now in the area of combinatorial optimization. The reason is that we do not know at the moment how to deal with the minimization global conditions like partitioning the sets of vertices into halves. The MIN-BISECTION problem arises naturally in several contexts ranging from combinatorial optimization to computational geometry and statistical physics [H97]. At the moment we do not have any approximation hardness result for MIN-BISECTION cf. [BK01], thus we cannot exclude a possibility of existence of a PTAS for that problem. On other hand the best known approximation factor for that problem is only $O(\log^2 n)$ [FK00].

Here we consider the metric version of that problem: we consider a finite set V of points which we call vertices together with a metric $d(.,.)$ on V and we ask for a partition of V into two equal parts such that the sum of the distances from the points of one part to the points in the other is minimized. It is easy to see that the metric MIN-BISECTIOM is NP-hard in exact setting even if restricted to weights 1 and 2. In this paper we prove somewhat surprisingly that the general metric MIN-BISECTION possesses a PTAS.

We draw on two lines of research to develop our algorithm: One is by now the well known methods of so-called exhaustive sampling for additive approximation for optimization problems such as MAX-CUT [AKK95], [F96], [GGR96], [FK96], [FK97], [AFKK01]. The second one connects previous papers on approximate algorithms for metric problems and weighted dense problems [FK01] and [FVK00].
We describe now some of the new ideas which we used especially in contrast to [FK01]. The main problem was the problem of sampling. Just as in [FK01] uniform sampling does not work, and we need to sample by picking each vertex with a probability proportional to the sum of its distances to the other vertices. This was circumvented in [FK01] by dividing each vertex into an appropriate number of "clones" and doing standard sampling on the set of clones. Then one could conclude easily from the fact that the clones of each fixed vertex go together in a maximum cut. This does not hold anymore for MIN-BISECTION (and also MAX-BISECTION) where we cannot use this cloning procedure. We circumvent this by *guessing* the placement of the outliers.

We display first the *NP-hardness* of metric MIN-BISECTION in exact setting.

Proposition 1. *Metric* MIN-BISECTION *is NP-hard in exact setting.*

Proof: We use the fact that metric MAX-CUT and, in fact, metric MAX-BISECTION problems are both NP-hard, even if restricted to the weights 1 and 2. (cf. [FK01], Theorem 1). We reduce from MAX-CUT following the reduction of [FK01]. Let OPT be the optimum of a MAX-CUT instance G of size n and OPT' be the optimum of a MAX-BISECTION instance H of size $2n$ to which G has been reduced. We have OPT'=2OPT+n^2. We construct now a complementary weighted graph H' by assigning weight 1 to all edges of H with weight 2, and weight 2 elsewhere. Let OPT" be the optimum of a metric MIN-BISECTION instance of H'. We have

$$\begin{aligned}\mathrm{OPT}'' &= 2((n/2)^2 - \mathrm{OPT}) + n^2 \\ &= \frac{n^2}{2} - 2\mathrm{OPT} + n^2 \\ &= \frac{3n^2}{2} - 2\mathrm{OPT}.\end{aligned}$$

Thus the exact computation of an optimum for metric MIN-BISECTION instances with weight 1 and 2 is NP-hard.

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2 Organization of the paper

The rest of the paper is organized as follows.
In Section 3, we formulate some metric lemmas which we need later. In Section 4, we give an algorithm for the Euclidean case and the analysis of its correctness. Finally, in Section 5, we construct a PTAS for the general metric MIN-BISECTION problem.

3 Preliminary results

Given a finite metric (V, d), we define

$$w_x = \sum_{y \in V} d(x, y)$$

for each $x \in V$, and

$$W = \sum_{x \in V} w_x.$$

Thus, W is twice the sum of all distances in V.

We define also for $U \subseteq V$,

$$W_U = \sum_{v \in U} w_v.$$

First, a couple of metric lemmas.

Lemma 1

$$d(v, u) \leq \frac{4 w_v w_u}{W}$$

Proof: See [FK01].

■

Lemma 2

$$\forall u \ \max_v d(u, v) \leq W/n$$

Proof: See [FK01].

■

The following lemma is crucial here. It shows that it suffices to obtain an additive approximation (within ϵW) to get a PTAS for metric MIN-BISECTION.

Lemma 3 *In the metric case, the optimal value of* MIN-BISECTION *satisfies* OPT $\geq W/5$.

Proof: Let $X = L \cup R$ be the optimal min bisection, of value OPT. Let $W = \sum_{X \times X} d(x, y)$, $W_L = \sum_{L \times L} d(x, y)$, and $W_R = \sum_{R \times R} d(x, y)$. Take 2 points x_1 and x_2 at random uniformly with replacement from L and 2 points x_3 and x_4 at random uniformly with replacement from R, and conside r the 6 edges of their induced subgraph. Then the contribution to the bisection is $a = d(x_1, x_3) + d(x_1, x_4) + d(x_2, x_3) + d(x_2, x_4)$, with expectation $4OPT/(n^2/4)$, and the contribution to $W_L + W_R$ is $d(x_1, x_2) + d(x_3, x_4)$, with expectation $(W_L + W_R)/(n^2/4)$, and satisfies:

$$\begin{aligned} d(x_1, x_2) &\leq d(x_1, x_3) + d(x_3, x_2) \\ d(x_1, x_2) &\leq d(x_1, x_4) + d(x_4, x_2) \\ d(x_3, x_4) &\leq d(x_3, x_1) + d(x_1, x_4) \\ d(x_3, x_4) &\leq d(x_3, x_2) + d(x_2, x_4) \\ d(x_1, x_2) + d(x_3, x_4) &\leq a \end{aligned}$$

Hence $W_L + W_R \leq 4OPT$, and so $W \leq 5OPT$.

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4 A Fixed Dimension Case

In the Euclidean case, when the dimension of the underlying space is fixed, a PTAS for MIN-BISECTION can be easily obtained. Here, we describe the PTAS for MIN-BISECTION in the plane. The cases of higher but fixed dimension are completely similar (replacing polar coordinates by spherical coordinates).

4.1 The Algorithm

The algorithm is the following.

Input: A set V of n points in the Euclidean plane.

1. Scale the problem so that the average interpoint distance is equal to 1.
2. Compute $g = \sum_{x \in V} x/n$, the center of gravity of V.
3. If $(d(x, g), \theta(x))$ denote the polar coordinates of x w.r. to g, define the domains

$$D_{r,k} = \left\{ x \in \mathrm{R}^2 : \begin{array}{l} \epsilon(1+\epsilon)^{r-1} \leq d(x, g) < \epsilon(1+\epsilon)^r \text{ and} \\ k\pi\epsilon \leq \theta(x) < (k+1)\pi\epsilon \end{array} \right\},$$

where $r \geq 1$ and $0 \leq k < 2\pi/\epsilon$. Let

$$D_0 = \{x \in \mathrm{R}^2 : d(x, g) < \epsilon\}.$$

4. Construct a point (multi)set V' obtained by replacing each element of $V \cap D_{r,k}$ by $y_{r,k}$, the point with polar coordinates $d(y_{r,k}, g) = \epsilon(1+\epsilon)^{r-1}$ and $\theta(y_{r,k}) = k\pi\epsilon$. Hence $y_{r,k}$ has multiplicity $m_{r,k}$ equal to the number of points in $V \cap D_{r,k}$. Moreover, each element of $V \cap D_0$ is replaced by g.

5. Let $s = 1 + \log_{1+\epsilon}(n/2\epsilon)$. Let $\omega_{r,k}$ denote the weighted distance from $y_{r,k}$ to X':

$$w_{r,k} = \sum_{0 \leq j \leq 2\pi/\epsilon} \sum_{0 \leq \ell \leq s} m_{j,l} d(y_{r,k}, y_{j,\ell}).$$

   Note that a partition $(L, X' \backslash L)$ of X' is defined by the set of pairs of integers $(p_{r,k}, q_{r,k})$ with $q_{r,k} = m_{r,k} - p_{r,k})$ where for each $0 \leq k < 2\pi/\epsilon$ and $0 \leq r \leq s$, $p_{r,k}$ denotes the number of points in $D_{r,k}$ which belong to L. We do exhaustive search on all the bisections corresponding to $p_{r,k}$ with $0 \leq p_{r,k} \leq m_{r,k}$ when $m_{r,k} \leq 1/\epsilon^2$, and with $p_{r,k} \in \{j\lfloor \epsilon^2 m_{r,k} \rfloor : 0 \leq j \leq 1/\epsilon^2 - 1\}$. for $m_{r,k} > 1/\epsilon^2$. We output the best bisection found.

Note that there are $O(\log n)$ domains $D_{r,k}$. Thus the exhaustive search tests at most $(1/\epsilon^2)^{O(\log n)} = n^{O(\log(1/\epsilon))}$ distinct bisections.

4.2 Analysis of Correctness

Let us analyse the effect of the restrictions of the sizes of the possible intersections of each domain with each side of the cut.
Let J denote the set of admissible pairs (r, k). Given an optimum bisection
OPT=$(p_{r,k}, m_{r,k} - p_{r,k})_{r \leq s, k \leq \pi/\epsilon}$ of V', we are guaranteed that our exhaustive search tests a bisection OPT'=$(p'_{r,k}, q'_{r,k})_{r \leq s, k \leq \pi/\epsilon}$ with $|p'_{r,k} - p_{r,k}| \leq \epsilon^2 m_{r,k}$. Denote by Q the set of pairs (r, k) for which the inequality $\epsilon m_{r,k} \leq p_{r,k} \leq (1-\epsilon) m_{r,k}$ is satisfied. Clearly, the $y_{r,k}$ for $(r, k) \notin Q$ contribute at most ϵW to the bisection. We have thus,

$$Val(\mathrm{OPT}') - Val(\mathrm{OPT}) \leq \sum_{(r,k) \in Q} \sum_{(s,\ell) \in Q} \left(p'_{r,k} q'_{s,\ell} - p_{r,k} q_{s,\ell}\right) d(y_{r,k}, y_{s,l}) + \epsilon W$$

For $(r, k) \in Q$, $(s, \ell) \in Q$ we have $|p'_{r,k} - p_{r,k}| \leq \epsilon p_{r,k}$, $|q'_{s,l} - q_{s,l}| \leq \epsilon q_{s,l}$, and so

$$\begin{aligned} Val(\mathrm{OPT}') - Val(\mathrm{OPT}) &\leq \sum_{(r,k) \in Q} \sum_{(s,\ell) \in Q} \left((1+\epsilon)^2 p_{r,k} q_{s,\ell} - p_{r,k} q_{s,\ell}\right) d(y_{r,k}, y_{s,l}) + \epsilon W \\ &\leq (1+3\epsilon) \sum_{(r,k) \in Q} \sum_{(s,\ell) \in Q} p_{r,k} q_{s,\ell} d(y_{r,k}, y_{s,l}) + \epsilon W \\ &\leq (1+3\epsilon)\mathrm{OPT} + \epsilon W \\ &\leq (1+8\epsilon)\mathrm{OPT}, \end{aligned}$$

the last because we know that OPT $\geq W/5$ by Lemma 3. The proof that the preliminary grouping of the vertices does not change the value of the optimum bisection by more than $O(\epsilon W)$ is similar to the proof given in [FK01] and is omitted. Thus we have a PTAS for MIN-BISECTION on the Euclidean plane.

5 A PTAS for the General Metric MIN-BISECTION

In this section we design and analyse a PTAS for the general metric MIN-BISECTION. Our algorithm builds on a sequence of papers [AKK95], [F96], [GGR96], [FVK00], and [FK01], and introduces a novel technique which combines biased metric sampling with some new hybrid placement and partitioning method.

Definition 1 *Consider a partition (L, R) of V. A multiset T of vertices with multiplicities $\mu(u)$, $u \in V$ is called (δ, ϵ)-representative with respect to P, if for every vertex v except perhaps for a subset of* **exceptional** *vertices of weight at most δW, we have*

$$\left| \frac{WD(V,t)}{|T|} \sum_{u \in T \cap L} \frac{\mu(u) d(u,v)}{w_u} - \sum_{u \in L} d(u,v) \right| \leq \epsilon w_v,$$

and

$$\left| \frac{WD(V,t)}{|T|} \sum_{u \in T \cap R} \frac{\mu(u) d(u,v)}{w_u} - \sum_{u \in R} d(u,v) \right| \leq \epsilon w_v.$$

A vertex which is not exceptional is called **normal**.

Our PTAS for metric MIN-BISECTION will make essential use of the following lemma.

Lemma 4 *Let t be any fixed integer ≥ 1. Let $U \subseteq V$, $W_U = \sum_{u \in U} w_u$. Let T be a random sample obtained by picking independently with replacement t points $u_i \in U$ with the probability distribution defined by $\Pr(u_i = u) = w_u / W$ $\forall u \in U$. (Thus our sample may be a multi-set.) Let $\mu(u)$ be the multiplicity of u in T. Let $v \in V$, and $D(v,T) = \sum_{u \in T \cap U} \frac{\mu(u) d(v,u)}{w_u}$. Then*

$$\Pr\left(\left| \frac{WD(v,T)}{t} - \sum_{u \in U} d(v,u) \right| \leq \epsilon w_v \right) \leq \frac{W_U}{\epsilon^2 t W} \tag{1}$$

Proof: We have:

$$D(v,T) = \sum_{i=1}^{t} Y_i$$

where the Y_i are pairwise independent and each distributed as Y_1 with

$$\Pr(Y_1 = \frac{d(v,u)}{w_u}) = \frac{w_u}{W} \ \forall u \in U,$$

and

$$\Pr(Y_1 = 0) = 1 - \frac{W_U}{W}.$$

We have clearly that $\mathbf{E} D(v,T) = \frac{t}{W} d(v,U)$, and

$$\begin{aligned} \mathbf{Var}(D(v,T)) &\leq t \sum_{u \in U} \frac{w_u}{W} \left(\frac{d(v,u)}{w_u} \right)^2 \\ &\leq \frac{t}{W} \sum_{u \in U} \frac{d(v,u)^2}{w_u} \\ &\leq \frac{t}{W^3} \sum_{u \in U} w_v^2 w_u, \end{aligned}$$

the last by using Lemma 1. Since $\sum_{u \in U} w_u = W_U$, this gives

$$\mathbf{Var}(D(v,T)) \leq \frac{t w_v^2 W_U}{W^3}$$

and

$$\mathbf{Var}\left(\frac{W D(v,T)}{t}\right) \leq \frac{w_v^2 W_U}{W t}.$$

Observing that $\mathbf{E}\left(\frac{WD(v,T)}{t}\right) = \sum_{u \in U} d(v,u)$, the assertion of the lemma follows by Chebyshev's inequality.

■

Let us choose $t = 20/\epsilon^5$ so that the probability in 1 is at most $\epsilon^3/20$, implying that the expectation of the number of vertices for which at least one of the inequalities in definition 1 does not hold is at most $\epsilon^3 n/10$. This implies in turn that the expectation of the weights of the corresponding vertices does not exceed $\epsilon^3 W/10$. Thus, using Markov inequality, this weight will not exceed $\epsilon^2 W$ with probability at least $1 - \epsilon/10$. This proves the following lemma.

Lemma 5 *Let T be a random sample of V with size $|T| = t$, defined as in Lemma 4 and let (L,R) be an arbitrary bipartition of V. If $t \geq 20/\epsilon^5$, then with probability at least $1-\epsilon/10$, T is (ϵ, ϵ^2)-representative with respect to (L,R) and moreover, the total weight of the exceptional vertices does not exceed $\epsilon^2 W$.*

Proof: See above.

■

Lemma 6 *Let V' denote the set of vertices of weight less than $\epsilon^2 W$. If V_j is a random subset of V' obtained by picking each vertex $v \in V$ with probability ϵ, then with probability at least $1-\epsilon^2$ we have $\sum_{v \in V_j} w_v \leq 2\epsilon W$. With probability ≥ 0.9 we have $\sum_j \sum_{u,v \in V_j} d(u,v) \leq 11\epsilon W$.*

Proof: (i) The sum $\sum_{v \in V_j} w_v$ is dominated by the product $\epsilon^2 W.B$ where B has the Binomial distribution $\mathrm{BIN}(n, \epsilon)$. The result follows by applying a Chernoff-Hoeffding Bound.
(ii) The sum $\sum_j \sum_{u,v \in V_j} d(u,v)$ has expectation bounded above by the sum $\frac{W}{n^2} \sum_j |V_j||V_j - 1| \leq 1.1\epsilon W$. The result follows by using Markov inequality.

■

We need the following lemma.

Lemma 7 *Let (L,R) be an optimum bisection of V Let $\ell = 1/\epsilon$ and define a partition $V_1, V_2, ..., V_\ell$ of V by placing each vertex in a randomly chosen V_j. With probability $1 - o(1)$, there exists a partition (A,B) whose cost is within an additive error at most ϵW from the optimum bisection and such that for each j it satisfies*

$$||A \cap V_j| - |B \cap V_j|| \leq 1. \tag{2}$$

Proof: Let $L_j = V_j \cap L, \;\; R_j = V_j \cap R$. For each j we do the following:

- If $|L_j| > |R_j|$, we set $\delta_j = \lfloor \frac{|L_j|-|R_j|}{2} \rfloor$ and we move from R to L δ_j vertices randomly chosen in L_j

- If $|L_j| < |R_j|$, we set $\delta_j = \lfloor \frac{|R_j|-|L_j|}{2} \rfloor$ and we move from L to R δ_j vertices randomly chosen in R_j.

Clearly, the resulting partition satisfies to 2. Let MV be the set of vertices whose positions have been changed according to the above rules and let Δ be the resulting loss in the objective function. Clearly,

$$\Delta \leq \sum_{x \in MV} w_x,$$

and so

$$\mathbf{E}(\Delta) \leq \sum_{x \in V} p_x w_x, \tag{3}$$

where p_x is the probability that x is moved. Fix now attention on a particular $x \in L$. (The case of $x \in R$ is exactly similar.) Assume without loss of generality that $x \in V_1$. We have that $|L_1| - 1$ has the binomial distribution with parameters $n/2 - 1$ and $p = \epsilon$. Also, $|R_1|$ has the binomial distribution with parameters $n/2$ and $p = \epsilon$. Chernoff-Hoeffding gives

$$\Pr(|L_1 - EL_1| \geq \sqrt{n \log n} \leq 2n^{-1/\epsilon}$$

and the analogue for $|R_1|$. Thus,

$$\Pr(|L_1 - EL_1| + |R_1 - ER_1| \geq 2\sqrt{n \log n}) \leq 4n^{-1/\epsilon}$$

For fixed $|L_1|$ and $|R_1|$, we have $p_x \leq \frac{|L_1 - EL_1| + |R_1 - ER_1|}{\epsilon n}$ and thus

$$\begin{aligned} p_x &\leq \frac{4}{\epsilon}\sqrt{\frac{\log n}{n}} + 4n^{-1/\epsilon} \\ &\leq \frac{5}{\epsilon}\sqrt{\frac{\log n}{n}} \end{aligned}$$

and using (3),

$$\begin{aligned} \mathbf{E}(\Delta) &\leq W \max_x p_x \\ &\leq \frac{5W}{\epsilon}\sqrt{\frac{\log n}{n}}. \end{aligned}$$

The lemma follows now by using Markov inequality. Note that the result proved is much stronger than the result claimed.

■

5.1 The algorithm

The algorithm takes as input a finite metric space (V, d). It makes a series of guesses and returns, when all these guesses are correct and with probability at least 3/4 a bisection of V whose cost is within $O(\epsilon W)$ from the optimum.

1. Compute vertex weights $w_v = \sum_u d(u, v)$ and total weight $W = \sum_v w_v$.
2. Let X denote the set of vertices with weight $> \epsilon^2 W$ and let $V' = V \backslash X$.

   Let $\ell = 1/\epsilon$ and define a partition $V_1, V_2, ..., V_\ell$ of V' by placing each vertex in a randomly chosen V_j.
3. Let $P_o = (L, R)$ be a bisection (L, R) with value at most ϵW from the optimum and with the property that it induces on each V_i a partition whose parts sizes differ by at most one. (The existence of such a partition is guaranteed by Lemma 7.) By exhaustive search, find the partition (X_L, X_R) of X induced by P_0. Let (L_j, R_j) be the partition of V_j induced by P_o. In the next phase the algorithm will construct inductively a sequence of "hybrid" partitions $P_0, P_1, ..., P_j, ..., P_\ell$ where the first hybrid is P_0, the last partition P_ℓ is the output, and such that, for each fixed j, P_j coincides with P_0 on each of the sets $V_{j+1}, V_{j+2}, ..., V_\ell$:
4. For each $j = 1, 2, \ldots, \ell$, do the following :
   (a) Let T_{j-1} denote a random multiset of V obtained by picking t times a vertex v of V according to the probabilities tw_v/W, $v \in V$, where t is defined as in Lemma 5.
   (b) By exhaustive search, guess the partition (T'_{j-1}, T''_{j-1}) induced on T_{j-1} by $(X_L, X_R), (A_1, B_1), \ldots, (A_{j-1}, B_{j-1}), (L_j, R_j), \ldots, (L_\ell, R_\ell)$.
   That is, classify the vertices of T_{j-1} which are in $X, V_1, V_2, \ldots, V_{j-1}$ according to the partition being built by the algorithm, and classify the remaining vertices of T_{j-1} according to the optimal partition guessed by exhaustive search.
   (c) For $v \in V_j$, let
   $$\hat{b}(v) = \sum_{u \in T'_{j-1}} \frac{d(u, v)}{w_u} - \sum_{u \in T''_{j-1}} \frac{d(u, v)}{w_u}.$$
   (d) Construct a partition (A_j, B_j) of V_j by placing the $|V_j|/2$ vertices with smallest value of $\hat{b}(v)$ in A_j and placing the other $|V_j|/2$ vertices in B_j.

   Let $A = \cup_j A_j$ and $B = \cup_j B_j$.
5. Output the best of the bisections (A, B) thus constructed.

5.2 The Analysis

Recall that for each $j \in \{0, ..., \ell\}$ P_j is the partition which agrees with the partitions $(A_1, B_1), ..., (A_j, B_j)$ constructed by the algorithm in $V_1, ..., V_j$, and which agrees with the optimal partition (L, R) in $V_{j+1}, ..., V_\ell$.

We will prove that when the algorithm correctly guesses for each j the partition (T_j', T_j'') induced on a random sample T_j by P_j, then the bisection (A, B) is optimal within at most $16\epsilon W$ with probability at least 3/4. The analysis will consist in showing that the increase of the objective function when changing one hybrid bisection into the next is small. We will need the following definition.

Definition 2 *Consider a partition $P = (L, R)$ of V. The unbalance of a vertex $v \in V$ with respect to P is the quantity*

$$\widehat{ub}(v) = \sum_{u \in L} d(u, v) - \sum_{u \in R} d(u, v).$$

Lemma 8 *If T_{j-1} is representative with respect to P_{j-1}, then* $\text{COST}(P_j) - \text{COST}(P_{j-1}) \leq 5\epsilon^2 W + \sum_{u,v \in V_j} d(u, v)$.

Before proving the lemma, let us first see how to use it to complete the analysis. By Lemma 5 the set T_{j-1} has probability at least $1 - \epsilon/10$ of being representative with respect to P_{j-1}. Thus, with probability at least $1 - \ell\epsilon/10 = 9/10$, T_{j-1} is representative for every j and Lemma 6 holds for every j. Summing over j, we then deduce that in that case:

$$\begin{aligned} \text{COST}((A, B)) - \text{OPT} &= \text{COST}(P_\ell) - \text{COST}(P_0) \\ &\leq 2\epsilon W + 5\epsilon W + \sum_j \sum_{u,v \in V_j} d(u, v) \\ &\leq 18\epsilon W \end{aligned}$$

the last by lemma 6 with probability at least 0.9. This implies with Lemma 3 a relative approximation ratio $1 + 90\epsilon$. To conclude the proof, it remains to verify that the result holds with probability at least 3/4 as claimed,when all the guesses are correct, The probability that the result does *not* hold is bounded above by the sum of:
- the probability that Lemma 7 does not hold which is $o(1)$
- the probability that at least one of the samples $T_1, T_2, ..., T_\ell$ is not (ϵ^2, ϵ)-representative which is bounded above by 1/10
- the probability that Lemma 6 does not hold for at least one j which is bounded above by 1/9.

The sum of these bounds is smaller than 0.25 and the claim follows

The running time is $2^{O(1/\epsilon^6)} n^2$ where the first factor accounts for the required number of exhaustive searches and n^2 is, within a constant factor, an upper bound for the number of operations needed for any fixed sequence of guesses. Hence, the algorithm is a PTAS for MIN-BISECTION on metric spaces.

We now proceed to prove Lemma 8.

Proof: [of Lemma 8.]
The only vertices which are classified differently in P_{j-1} and in P_j are vertices in V_j : say, x vertices are in the left side of P_{j-1} and in the right side of P_j, and the same number x of vertices are in the left side of P_j and in the right side of P_{j-1}. Pair up these vertices in a matching M. For each such pair (u, v), such that P_{j-1} places v on the right side and u on the left side, let $P_{j-1}(u, v)$ denote the partition obtained from P_{j-1} by switching the sides of vertices u and v. Note that by definition of the algorithm, $\hat{b}(u) \geq \hat{b}(v)$.
Note that the overall probability that for each j, T_j is representative is at least 9/10, so we can assume that this is the case. Then,

$$
\begin{aligned}
&\text{COST}(P_j(u,v)) - \text{COST}(P_{j-1}) \\
&\leq \widehat{\text{ub}}(u) - \widehat{\text{ub}}(v) \\
&= (\widehat{\text{ub}}(u) - \widehat{\text{ub}}(v)) - \frac{W}{t}(\hat{b}(u) - \hat{b}(v)) + \frac{W}{t}(\hat{b}(u) - \hat{b}(v)) \\
&\leq (\widehat{\text{ub}}(u) - \widehat{\text{ub}}(v)) - \frac{W}{t}(\hat{b}(u) - \hat{b}(v)) \\
&\leq |\widehat{\text{ub}}(u) - \frac{W}{t}\hat{b}(u)| + |\widehat{\text{ub}}(v) - \frac{W}{t}\hat{b}(v)|.
\end{aligned}
$$

There are two cases.
(i) If u and v are normal, then we use the upper bounds $|\widehat{\text{ub}}(u) - \frac{W}{t}\hat{b}(u)| \leq \epsilon w_u$, $|\widehat{\text{ub}}(v) - \frac{W}{t}\hat{b}(v)| \leq \epsilon w_v$.
(ii) For the total contribution of the exceptional vertices, we use the overall bound $\epsilon^2 W$ of Lemma 5. Also

$$
\text{COST}(P_j) - \text{COST}(P_{j-1}) \leq \\
\sum_{(u,v)\in M} (\text{COST}(P_{j-1}) - \text{COST}(P_{j-1}(u,v))) + \sum_{u,v\in V_j} d(u,v).
$$

Thus,

$$
\begin{aligned}
\text{COST}(P_j) - \text{COST}(P_{j-1}) &\leq 2\epsilon \sum_{u\in V_j} w_u + \epsilon^2 W + \sum_{u,v\in V_j} d(u,v) \\
&\leq 5\epsilon^2 W + \sum_{u,v\in V_j} d(u,v)
\end{aligned}
$$

the last by using Lemma 6.

■

This completes the correctness proof of our PTAS for the general metric MIN-BISECTION.

Acknowledgements. We thank Mark Jerrum, Alan Frieze, and Ravi Kannan for stimulating remarks and discussions.

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