Approximation Algorithms for MAX-BISECTION on Low Degree Regular Graphs and Planar Graphs

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Abstract

The max-bisection problem is to find a partition of the vertices of a graph into two equal size subsets that maximizes the number of edges with endpoints in both subsets.

We obtain new improved approximation ratios for the max-bisection problem on the low degree k-regular graphs for $3 \leq k \leq 8$, by deriving some improved transformations from a maximum cut into a maximum bisection partition. In the case of three regular graphs we obtain an approximation ratio of 0.847, and in the case of four and five regular graphs, aproximation ratios of 0.805, and 0.812, respectively.

We also present the first polynomial time approximation scheme for the max-bisection problem for planar graphs of a sublinear degree.

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1 Introduction

The max-bisection problem, i.e., the problem of finding a halving of the vertex set of a graph that maximizes the number of edges across the partition, is one of the basic combinatorial optimization problem.

Frieze and Jerrum give the following example of its application [6]: there are m people, each of whom selects two activities among n ones. Assuming n to be even, split the activities evenly between two time slots in order to maximize the number of people who participate in both their activities.

The best known approximation algorithms for max-bisection yield a solution whose size is at least 0.701 times the optimum Halperin and Zwick [9]. It is still an open problem whether the max-bisection problem is hard to approximate (or even to compute exactly) on the planar graphs.

For dense graphs, Arora, Karger and Karpinski give polynomial time approximation schemes for max- and min-bisection in [1]. Recently, Feige, Karpinski and Langberg [4] have obtained sharp lower bounds on the ratio between the sizes of max bisection and max cut for regular graphs. They have also shown the bounds to be tight already for constant degree regular graphs [4]. By combining them with the known polynomial time approximation algorithms for max cut they have obtained an approximation ratio of 0.795 for max bisection (i.e., a polynomial time algorithm for max bisection of regular graphs producing a solution of size at least 0.795 times the optimum).

In this paper, for low degrees k, $3 \leq k \leq 8$, we derive sharper lower bounds on the ratio between the sizes of max bisection and max cut for k-regular graphs than those proven in [4] for constant degree regular graphs. In effect, we can derive improved approximation ratios for max bisection for k-regular graphs where $3 \leq k \leq 8$. In particular, we obtain the following approximation ratios: 0.847 for $k = 3$, 0.812 for $k = 5$, 0.803 for $k = 7$, and 0.805 for $k = 4, 6, 8$.

The max bisection problem can be seen as the max cut problem with an additional requirement on the equal size of the two subsets in the two partition. Because of this requirement the approximability status of max bisection is more evolved than that of max cut. This situation extends also to special graph classes, e.g., planar graphs. Planar max cut is known to admit an exact polynomial time algorithm [8] whereas the complexity and approximability status of planar max bisection is totally open. This contrasts with the case of many planar graph problems which are known to admit polynomial time approximation schemes by falling into Khanna-Motwani's syntactic framework [10].

Our second main result is the first polynomial time approximation scheme (PTAS) for the max bisection problem restricted to planar graphs of the sublinear maximum degree. It is obtained by efficient transformation from the max cut produced by an exact polynomial time algorithm of [8] into a bisection with a close size.

2 Preliminaries

We start here with some basic notions used throughout the paper.

Definition 2.1 *A real number α is said to be an approximation ratio for a maximization problem, or equivalently the problem is said to be* approximable within a ratio α, *if there is a polynomial time algorithm for the problem which always produces a solution of size at least α times the optimum. If a problem is approximable for arbitrary $\alpha < 1$ then it is said to admit a* polynomial time approximation scheme *a (PTAS for short).*

We formulate now the underlying optimization problems of max cut and max bisection.

Definition 2.2 *A partition of a set of vertices of an undirected graph G into two sets X, Y is called a* cut *of G and is denoted by (X, Y). A partition is a* bisection *if the cardinalities of X and Y are equal. The edges of G with one endpoint in X and the other in Y are said to be cut by a partition. A partition is called a* max cut *of G if it maximizes the number of cut edges. The partition is called a* max bisection *of G if it is a bisection of G maximizing the number of cut edges. The* max-cut problem *is to find a max cut of a graph. Analogously, the* max-bisection problem *is to find a max bisection of a graph.*

The following simple lemmas will be used in the next section.

Lemma 2.1 *Let (X, Y) be a max cut of a graph G. For any vertex v in Y, v has at least as many neighbors in X as in Y.*

Proof: Suppose otherwise. Then, the partition $(X \cup \{v\}, Y \setminus \{v\})$ would cut more edges than (X, Y). □

Definition 2.3 *A max cut (X, Y) of a graph G is said to be* maximally balanced *if it cannot be transformed into another max cut (X', Y') of G satisfying $||X'| - |Y'|| < ||X| - |Y||$ by moving a single vertex from X to Y or* vice versa.

The next lemma exhibits a useful property of a maximally balanced max cut.

Lemma 2.2 *Let (X, Y) be a maximally balanced max cut of a graph G where $|X| < |Y|$. For any vertex v in Y, v has more neighbors in X than in Y.*

Proof: If v had at most as many neighbors in X as in Y then by moving it to X we would obtain another max cut (X', Y') satisfying $||X'| - |Y'|| < ||X| - |Y||$; a contradiction. □

A max cut can be easily transformed into a maximally balanced max cut by the following lemma.

Lemma 2.3 *A max cut of a graph can be transformed into a maximally balanced max cut of the graph in linear time.*

Proof: We may assume w.l.o.g that the input max cut (X, Y) is not maximally balanced and $|X| < |Y|$. For each vertex v in Y as long as the size of the current Y is not equal to that of the current X we check whether or not it is possible to move v to the current X without decreasing the number of cut edges. If so, we move v to the current X.

The resulting max cut is maximally balanced since either the size of the final Y is equal to that of the final X or none of the vertices in the final Y can be moved to the final X without decreasing the number of cut edges. The latter follows from the observation that for each vertex v in the final Y the number of its neighbors in the final Y cannot exceed that in the original Y. So, if v could not change the side before, it cannot do it now.

Checking the vertices in Y takes linear time. □

3 Approximation of the max-bisection problem for low degree regular graphs

In this section we derive new lower bounds on the ratio between the size of max bisection and that of max cut for low degree regular graphs following the approach of [5]. By combining them with known polynomial time approximation algorithms for max cut on regular graphs we improve the known approximation ratios for low degree regular graphs substantially.

The following lemma is immediate.

Lemma 3.1 *For a positive integer k, let (X, Y) be a max cut of a k-regular graph G, and let E be the set of edges of G cut by (X, Y). Next, for $i = 0, ..., \lfloor (k-1)/2 \rfloor$, let Y_i be the set of vertices in Y with exactly i neighbors in Y. The inequalities $|X| \geq |E|/k$ and $\sum_{i=0}^{\lfloor (k-1)/2 \rfloor} (k-i)|Y_i| \leq |E|$ hold.*

Theorem 3.1 *For $k = 3, 4, 6, 8$ any max cut (X, Y) of a k-regular graph G can be transformed in linear time into a bisection of G cutting no less than $\frac{11}{12}$ of the number of edges cut by (X, Y). For $k = 5, 7$ any max cut (X, Y) of a k-regular graph G can be transformed in linear time into a bisection of G cutting no less than $\frac{37}{40}$ and $\frac{64}{70}$, respectively, of the number of edges cut by (X, Y).*

Proof: Assume the notation of Lemma 3.1, and w.l.o.g. $|X| < |Y|$. Also, we may assume w.l.o.g that for even k the max cut is maximally balanced since otherwise we can transform it to such a max cut in linear time by Lemma 2.3. Let l be the number of vertices in Y necessary to move to X in order to transform (X, Y) into a bisection of G.

By Lemmata 2.1, 2.2, 3.1, $l \leq \frac{1}{2}(\sum_{i=0}^{\lfloor (k-1)/2 \rfloor} |Y_i| - |E|/k)$. Consequently, we have $l \leq \frac{1}{2k}(k\sum_{i=0}^{\lfloor (k-1)/2 \rfloor} |Y_i| - |E|)$ which by Lemma 3.1 yields $l \leq \frac{1}{2k}(\sum_{i=1}^{\lfloor (k-1)/2 \rfloor} i|Y_i|)$.

Suppose we can pick an independent set I of vertices in Y composed of disjoint subsets I_i, $i = 1, ..., \lfloor k/2 \rfloor$, respectively containing at least $\frac{i|Y_i|}{2k}$ vertices in $\bigcup_{m=i}^{\lfloor (k-1)/2 \rfloor} Y_m$ (*). Then, by moving l vertices from I to X we obtain a bisection cutting at least $|E| - \frac{1}{2k}\sum_{i=1}^{\lfloor (k-1)/2 \rfloor}((k-i)-i)i|Y_i|$ edges. The latter sum is $|Y_1|$ for $k = 3$, $2|Y_1|$ for $k = 4$, $3|Y_1| + 2|Y_2|$ for $k = 5$, $4|Y_1| + 4|Y_2|$ for $k = 6$, $5|Y_1| + 6|Y_2| + 3|Y_3|$ for $k = 7$, and $6|Y_1| + 8|Y_2| + 6|Y_3|$ for $k = 8$. By Lemma 3.1, it can be bounded from above by $\frac{|E|}{2}$ for $k = 3$, $\frac{2|E|}{3}$ for $k = 4$, $\frac{3}{4}|E|$ for $k = 5$, $|E|$ for $k = 6$, $\frac{6}{5}|E|$ for $k = 7$, and $\frac{4}{3}|E|$ for $k = 8$. Thus, for $k = 3, 4, 5, 6, 7, 8$ the resulting bisection cuts at least $\frac{11}{12}|E|$, $\frac{11}{12}|E|$, $\frac{37}{40}|E|$, $\frac{11}{12}|E|$, $\frac{64}{70}|E|$ and $\frac{11}{12}|E|$ edges, respectively.

To pick such an independent set I, we may w.l.o.g. restrict ourselves to the subgraph G' of G induced by Y.

For $k = 3, 4$ we have $I = I_1$ and the condition (*) requires at most $|I_1| \geq \frac{1}{6}|Y_1|$. We can trivially pick an independent subset of Y_1 whose size is at least $\frac{1}{2}|Y_1|$ by avoiding picking both endpoints of any isolated edge in G'.

For $k = 5, 6$, it is sufficient if I_1 contains at least $\frac{1}{10}|Y_1|$ vertices in $Y_1 \cup Y_2$ whereas I_2 contains at least $\frac{1}{5}|Y_2|$ vertices in Y_2, by (*). Observe that G' consists of isolated vertices, paths, and possibly cycles. We can easily form the independent set I_1 consisting of $\frac{1}{10}$ of the vertices in Y_1 by accounting to it the isolated vertices in G', further if this is not enough, single endpoints of isolated edges in G', and if this is still not enough, single endpoints of paths of length greater than one in G'. Let P be the set of the latter paths in G'. Note that I_1 eliminates at most $|P|/10$ of neighbors of degree two in G', and all vertices in Y_2 are inner vertices on paths in P. Therefore, since $\frac{1}{2}(|Y_2| - |P|/10) \geq \frac{1}{5}|Y_2|$ by $|P| \leq |Y_2|$, we can easily form the required independent set I_2 by taking at least every second vertex not adjacent to I_1 on the paths in P.

For $k = 7, 8$ the following three conditions are jointly not weaker than (*).

1. I_1 should have at least $\frac{|Y_1|}{14}$ vertices in $Y_3 \cup Y_2 \cup Y_1$.
2. I_2 should contain at least $\frac{2|Y_2|}{14}$ vertices in $Y_3 \cup Y_2$, and
3. I_3 should contain at least $\frac{3|Y_3|}{14}$ vertices in Y_3.

While there is a vertex v of degree three in G', we iterate the following step: augment I by v and delete v and its neighbors.

Let I_3' be the resulting independent set of vertices of degree three in the original G'. Next, let s be the number of vertices of degree one or two in the original G' that have at least one neighbor in I_3'. The inequality $|Y_3 - I_3'| + s \leq 3|I_3'|$ immediately follows from the way of picking I_3'. This yields $|Y_3| + s \leq |I_3'| + |Y_3 - I_3'| + s \leq 4|I_3'|$. Hence, the surplus of vertices of degree three in I_3' with respect to the requirement (3) on I_3 is at least $|I_3'| - \frac{3|Y_3|}{14} \geq |I_3'| - \frac{3}{14}(4k_1 - s)$ $\geq \frac{2}{14}k_1 + \frac{3}{14}s > \frac{s}{7}$. It yields a sufficient number of vertices in I to fulfill the requirements (1) and (2) proportionally for the aforementioned s vertices of degree one and two. We can pick up the appropriate proportions of independent vertices required in (1) and (2) among the remaining vertices of degree one and two in the original graph G' analogously as in the case $k = 5, 6$. □

Corollary 3.1 *Let $\alpha = 0.878$ be the approximation ratio achievable for the max-cut problem [7], and let $\beta = 0.924$ be the approximation ratio achievable for the max-cut problem for three-regular graphs [4]. The max-bisection problem is approximable within a ratio of $\frac{11\beta}{12} = 0.847$ for three-regular graphs, within a ratio of $\frac{37\alpha}{40} = 0.812$ for five-regular graphs , within a ratio of $\frac{64\alpha}{70} = 0.803$ for seven-regular graphs, and within a ratio of $\frac{11\alpha}{12} = 0.805$ for four, six and eight-regular graphs.*

4 PTAS for max bisection on planar graphs

The requirements of the equal size of the vertex subsets in a two partition yielding a max bisection makes the max-bisection problem hardly expressible as a maximum planar satisfiability formula. For this reason we cannot directly apply Khanna-Motwani's [10] syntactic framework yielding PTAS for the planar restrictions of several basic graph problems including max cut. Instead, we choose to produce a max cut of the input graph by running the exact polynomial time algorithm of Hadlock [8] and then to transform it into a bisection of a close size. Our transformation is based on the following known fact on edge separability of planar graphs.

Fact 1[2]. *Let G be an n-vertex planar graph of maximum degree d. G has an edge separator of size $O(\sqrt{2dn})$, i.e., a set of edges whose removal disconnects G into two subgraphs none of which has more than two thirds of the vertices of G. Furthermore, such an edge set can be found in time $O(n)$.*

Analogously as Theorem 3 in [11] is obtained from the original Lipton-Tarjan vertex planar separator theorem, we obtain the following useful theorem from Fact 1.

Theorem 4.1 *Let G be an n-vertex planar graph of maximum degree d with nonnegative vertex costs summing to no more than one and let $0 < \epsilon \leq 1$. Then there is some set C of $O(\sqrt{dn/\epsilon})$ edges whose removal leaves G with no connected component of cost exceeding ϵ. Furthermore the set C can be found in time $O(n \log n)$.*

We prove that the following algorithm yields a PTAS for planar graphs of a sublinear degree.

Algorithm Bisection

input: a planar graph G on n vertices

output: a bisection of G

1. Find a maximum cut of G. Let $V_l,\ V_r$ be the two subsets of V inducing the cut.
2. Apply Theorem 4.1 with $\epsilon = k(n)/n$ to find a set C of edges of size $O(n\sqrt{d/k(n)})$ whose removal leaves no connected component with more than $k(n)$ vertices.
3. Set $V_1,\ V_2$ to empty sets.

   for each connected component C_i of G resulting from removing C from G **do**

   if $|V_1| > |V_2|$ and $|C_i \cap V_l| < |C_i \cap V_r|$ or $|V_1| \leq |V_2|$ and $|C_i \cap V_l| \geq |C_i \cap V_r|$ **then** augment V_1 by $C_i \cap V_l$ and V_2 by $C_i \cap V_r$ **else** augment V_1 by $C_i \cap V_r$ and V_2 by $C_i \cap V_l$
4. **if** $|V_1| > |V_2|$ **then** sort V_1 by vertex degree in non-decreasing order and augment V_2 by the first $(|V_1| - |V_2|)/2$ vertices in V_1 **else** sort V_2 by vertex degree in non-decreasing order and augment V_2 by the first $(|V_2| - |V_1|)/2$ vertices in V_2
5. Output (V_1, V_2).

By the fourth step we obtain:

Lemma 4.1 *Algorithm Bisection produces a bisection of G.*

Lemma 4.2 *The size of the cut produced by Algorithm Bisection is at least the size of maximum cut of G decreased by $O(n\sqrt{d/k(n)} + k(n))$.*

Proof: Since each component C_i has size not exceeding $k(n)$, the difference between the sizes of V_1 and V_2 never exceeds $k(n)$ by induction on the number of iterations of the block in Step 3. The cut produced by Algorithm Bisection includes in particular all the edges belonging to the maximum cut produced in Step 1 that are outside C and are not incident to the at most $k(n)/2$ vertices moved from V_1 to V_2 or *vice versa* in Step 4. By planarity of G implying the $O(1)$ average vertex degree and the choice of the at most $k(n)/2$ vertices, there are only $O(k(n))$ edges incident to them. $\square$

Lemma 4.3 *Algorithm Bisection can be implemented in polynomial time.*

Proof: Step 1 can be implemented in polynomial time by [8]. Step 2 takes $O(n \log n)$ time by [2]. Steps 3, 4 can be easily implemented in time $O(n \log n)$ by using basic data structures. □

Theorem 4.2 *The max-bisection problem for connected planar graphs of degree $o(n)$ admits a PTAS.*

Proof: Let G be a connected planar graph of maximum degree d. Set $k(n)$ in Algorithm Bisection to βd where β is a rational to be specified later. By Lemma 4.2, Algorithm Bisection produces in polynomial time a bisection of G whose size is at most that of max cut of G decreased by $\beta^{-1/2}n + o(\beta n)$. It is sufficient to observe that max cut of G has size $\Omega(n)$ and choose a sufficiently large β. □

5 Final remark

Note that for an instance of a star graphs, the ratio between the size of max bisection and that of max cut can be arbitrarily close to $\frac{1}{2}$. For that reason, our approach of transforming a max cut of a planar graph into its bisection of close size cannot directly work for a graph has a linear maximum degree. An interesting open problem remains to extend our result to arbitrary planar graphs.

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