Polynomial Time Approximation Schemes for Some Dense Instances of NP-Hard Optimization Problems (Extended Version)*

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Abstract

We survey recent results on the existence of polynomial time approximation schemes for some dense instances of NP-hard combinatorial optimization problems. We indicate further some inherent limits for existence of such schemes for some other dense instances of optimization problems. We also go beyond the dense optimization problems and show how other approximation problems can be solved by using dense techniques.

1 Introduction

The computational efficiency of approximating different *NP-hard optimization* problems varies a great deal. We know by now, that unless P=NP, some problems, such as CLIQUE cannot be approximated in polynomial time even within a factor $n^{1-\epsilon}$ for any $\epsilon > 0$ (cf. Håstad [H96]). Some other problems like MAX-CUT (cf. Goemans and Williamson [GW94]) or STEINER TREE (cf. Karpinski and Zelikovsky [KZ97a]), can be approximated to within some small fixed constant

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factor. Till recently only a very few optimization problems were known to have polynomial time approximation schemes (PTAS), approximating to within arbitrary small constant factors. In a recent breaktrough Arora [A96] shown existence of PTASs for Euclidean TSP and a number of other geometric problems.

Some of the approximation algorithms with small approximation ratios achieve also good practical performances, like some cases of STEINER TREE problems (cf. [KZ97a]), some other algorithms do not yield yet efficient practical solutions for dealing with optimization problems.

In this paper we are concerned with the problem of efficient approximability of the *dense instances* of *NP-hard* optimization problems.

Starting in 1995, the first polynomial time approximation schemes have been designed *for such problems* in Arora, Karger and Karpinski [AKK95], Fernandez de la Vega [FdV96], Arora, Frieze and Kaplan [AFK96], and Karpinski and Zelikovsky [KZ97b]. Later on, Goldreich, Goldwasser and Ron [GGR96], and Frieze and Kannan [FK97] gave a constant sample size approximation schemes for some dense optimization problems. Fernandez de la Vega and Karpinski [FdVK97] gave also the first polynomial time approximability characterization for dense *weighted* instances of NP-hard problems; same authors [FdVKa98], and also Fotakis and Spirakis [FS98] proved approximation hardness of dense (1,2)-TSP problem.

This development was in contrast to the fact that the existence of such schemes for general instances of all the above mentioned problems would imply that P=NP, by results of Arora, Lund, Motwani, Sudan and Szegedy [ALMSS92].

The development above was followed by the study of the dense covering problems in Karpinski and Zelikovsky [KZ97b], and the dense bandwidth minimization problems, Karpinski, Wirtgen and Zelikovsky [KWZ97]; as well as metric instances of MAX-CUT, Fernandez de la Vega and Kenyon [FdVKe98].

It is also a very interesting artifact that the recent successes in design of the polynomial time approximation schemes for dense optimization problems parallel the successes of the past attacks on dense approximate counting problems, Broder [B86], Jerrum and Sinclair [JS89], Dyer, Frieze, Jerrum [DFJ94], and Alon, Frieze and Welsh [AFW95].

2 MAX-SNP and Dense MAX-SNP Classes, and BEYOND

We consider in this Section the *dense* instances of the MAX-SNP class of optimization problems introduced by Papadimitriou and Yannakakis [PY91]. MAX-SNP class contains constraint-satisfaction problems, where the *constraints* are definable by quantifier-free propositional formulas.

We recall:

Definition: A (*maximization*) *problem* A is in MAX-SNP if there exists a sequence of relation symbols $G_1, \ldots, G_m$, a relation symbol S, and a quantifier-free formula $\Phi(G_1, \ldots, G_m, S, x_1, \ldots, x_k)$, with x_i variables, such that the following is true:

1. there is a polynomial time algorithm that for any given instance I of the problem A produces a set $\mathcal{V}$ and a sequence of relations $G_1^{\mathcal{V}}, \ldots, G_m^{\mathcal{V}}$ over $\mathcal{V}$ ($G_i^{\mathcal{V}}$ preserve the *arity* of G_i);

2. The value of the optimum solution $OPT(I)$ of A on instance I, satisfies

$$OPT(I) = \underset{S^{\mathcal{V}}}{\mathrm{MAX}} \left\{ \left| \{(x_1, \ldots, x_k) \in \mathcal{V}^k \middle| \Phi(G_1^{\mathcal{V}}, \ldots, G_m^{\mathcal{V}}, S^{\mathcal{V}}, x_1, \ldots, x_k) = TRUE\} \right| \right\}$$

   for $S^{\mathcal{V}}$ the relation over $\mathcal{V}$ of the same arity as S.

Example: MAX-CUT (cf. [GJ79], [P94]) is in MAX-SNP, since its optimum solution can be written as

$$\underset{S \subseteq \mathcal{V}}{\mathrm{MAX}} \left\{ \mid \{(x,y) \mid (G(x,y) \vee G(y,x) \wedge S(x) \wedge \neg S(y))\} \mid \right\}$$

for V the set of vertices of the graph, $G(x,y)$ its adjacency relation, and S a unary relation describing the *one side* of the cut.

For the notions of MAX-SNP-*completeness*, and MAX-SNP-*hardness* see [P94], and [AL97].

We define next the problem MAX-k-FUNCTION-SAT for some fixed integer k. MAX-k-FUNCTION-SAT has as an input m boolean functions $f_1, f_2, \ldots, f_m$

in n variables, and each f_i depends only on k variables. The problem is to find an assignment to the variables as to satisfy as many f_i's as possible.

It is known that every problem A from MAX-SNP can be viewed as a MAX-k-FUNCTION-SAT problem for a fixed k (cf. [P94]). Following [AKK95] we call an instance of a MAX-SNP problem *dense* if the corresponding instance of MAX-k-FUNCTION-SAT has $\Omega(n^k)$ functions.

Given an *optimization problem* A, a (meta) algorithm $\mathcal{A}$ is called a *polynomial time approximation scheme* (PTAS) if for every fixed $\epsilon > 0$, $\mathcal{A}$ is a *polynomial* time algorithm with *approximation ratio* $1+\epsilon$ (meaning $\mathcal{A}$ outputs a solution S to every instance I of A such that $\text{MAX}\left\{\frac{S}{OPT(I)}, \frac{OPT(I)}{S}\right\} \leq 1+\epsilon$, for OPT(I) the optimal solution, and the running time of $\mathcal{A}$ is polynomial in the size of I).

Only a very few problems, such as KNAPSACK [IK75], and BIN PACKING [FL81], [KK82], were till recently known to have PTASs.

In Arora, Karger and Karpinski [AKK95] the following general result on the existence of PTASs was proven.

Proposition 1. ([AKK95]) Dense MAX-SNP problems have PTASs.

The proof method involves the representation of MAX-k-FUNCTION-SAT by *smooth degree–k integer* programs, and the general result on approximating such programs (cf. [AKK95]).

Below is the list of problems were the smooth integer programs can be applied directly to obtain the PTASs. (We call a graph *dense* if it has $\Theta(n^2)$ edges, a hypergraph of dimension d is dense if it does have $\Theta(n^k)$ edges.)

- MAX-CUT: For a given graph, partition its vertices into two sets so as to maximize the number of edges between them.
- MAX-DCUT: The directed version of the MAX-CUT.
- MAX-HYPERCUT(d): A generalization of MAX-CUT to hypergraphs of dimension d (an edge is considered in a cut if it has at least one vertex on each side).
- DENSE-k-SUBGRAPH: Given a graph, find a subset of k vertices that induces a graph with the most edges (cf. [KP93]).

Following [AKK95], we have

Proposition 2. ([AKK95]) Dense instances of the following problems have PTASs:
MAX-CUT, MAX-DCUT, MAX-HYPERCUT(d), and DENSE-k-SUBGRAPH for $k = \Omega(n)$.

In what follows we call a graph G *everywhere dense* if its minimum degree is $\Omega(n)$. We consider *everywhere dense* instances of three further problems.

- SEPARATOR: Given a graph, partition its vertices into two sets, each with at least $\frac{1}{3}$ of the vertices, so as to minimize the number of edges between them.
- BISECTION: Given a graph, partition its vertices into two equal halves so as to minimize the number of edges between them.
- MIN-k-CUT: Given a graph with n vertices, and k *source* vertices, partition its vertices into k groups such that (1) each group contains one source, and (2) the number of edges between different groups is *minimized.*

Consider a graph with a *minimum* degree δn, and let c denote the capacity of its minimum bisection. The PTAS for BISECTION of [AKK95] consists of two algorithms, one of which is a PTAS when $c \geq \alpha n^2$, and the other when $c < \alpha n^2$ for α a small constant. The algorithm for $c \geq \alpha n^2$ uses the above mentioned method for approximating smooth integer programs. For the case $c < \alpha n^2$ we use the fact that in a *minimum bisection*, there must be one side whose every vertex has at most half of its neighbors on the other side, and construct a randomized exhaustive *correction* sample algorithm. The algorithm can be also easily derandomized (cf. [AKK95]). Similar PTASs work for the SEPARATOR and MIN-k-CUT problems.

Proposition 3. ([AKK95]) Everywhere dense instances of the following problems have PTASs: BISECTION, SEPARATOR and MIN-k-CUT.

Fernandez de la Vega [FdV96] has independently developed a PTAS for everywhere dense instances of MAX-CUT problem. His algorithm does not appear to generalize though to the other problems listed above.

Arora, Frieze, and Kaplan [AFK96] constructed a new rounding procedure for the quadratic assignment problem and used it to obtain PTASs on the dense instances of the NP-hard problems like QUADRATIC-ASSIGNMENT,

MIN-LINEAR-ARRANGEMENT, d-DIMENSIONAL-ARRANGEMENT, BETWEENESS, and MIN-CUT-LINEAR-ARRANGEMENT (cf. [AFK96]).

In the other development the Regularity Lemma of Szemerédi was used to obtain more efficient PTAS for the above problems (cf. Frieze and Kannan [FK96]). Using also independent methods Goldreich, Goldwasser and Ron [GGR96], and Frieze and Kannan [FK97] gave constant time approximation schemes for some dense problems in the oracle model of computation.

Very recently Bazgan and Fernandez de la Vega [BFdV99] designed the first PTAS for everywhere dense MIN-2SAT and MIN-EQUIVALENCE problems using techniques similar to that of a BISECTION technique of [AKK95].

3 Dense Covering Problems

We turn now to the three dense covering problems: SET COVER, STEINER TREE, and VERTEX COVER (cf. [H97], [AL97]). They do not fall into the dense MAX-SNP class definition of section 2 (VERTEX COVER is in MAX-SNP only if the degree of the graph is bounded.)

- SET COVER: Given a finite set X and a family of its subsets $\mathcal{P}$, find a minimum size subfamily $\mathcal{M}$ of $\mathcal{P}$ such that $X \subseteq \bigcup \mathcal{M}$.

  We call an instance of SET COVER $(X = \{x_1, \ldots x_n\}, \mathcal{P} = \{p_1, \ldots, p_m\})$ *ϵ-dense* (for $\epsilon > 0$) if every element of X belongs to at least ϵm sets from $\mathcal{P}$. (The instances of SET COVER are called *dense* if they are ϵ-dense for some $\epsilon > 0$. We call SET COVER restricted to dense instances a dense SET COVER accordingly.)

- STEINER TREE: Given a connected graph G and a set of its distinguished (terminal) vertices S. Find a minimum size tree within G that spans all distinguished vertices from S.

  We call an instance $G = (V, E)$ of the STEINER TREE problem *ϵ-dense* if every distinguished terminal vertex is adjacent to at least $\epsilon \cdot \mid V \setminus S \mid$ nonterminals.

- VERTEX COVER: Given a graph G, find a minimum size vertex set X of G which covers all edges of G (i.e. at least one endpoint of any edge belongs to X).

We start with the dense SET COVER problem. The general SET COVER was proven recently to have a threshold $(1 - o(1)) \ln n$ for any olynomial time approximation (cf. Feige [F96]), which in fact is matching asymptotically the approximation ratio by the well known greedy heuristic algorithm.

It is shown in Karpinski and Zelikovsky [KZ97b] that the greedy heuristic algorithm can be applied more efficiently towards ther dense SET COVER.

Proposition 4. ([KZ97b]) For any constant $c > 0$ and any $\epsilon > 0$, there is a polynomial time approximation algorithm for the ϵ-dense SET COVER with the approximation ratio $c \cdot \log n$.

Interestingly, we cannot expect on the lower bound side of the dense SET COVER, its NP-hardness, as the results of Papadimitriou and Yannakakis [PY96] imply.

Proposition 5. ([KZ97b]) Unless NP $\subseteq$ DTIME $[n^{\log n}]$, the dense SET COVER is not NP-hard.

We conjecture that dense SET COVER cannot be approximated to within a constant factor.

Conjecture 1. The dense SET COVER cannot be approximated in polynomial time to within a constant approximation ratio.

The second problem we discuss in this section is the ϵ-dense STEINER TREE problem. We note first in passing that for $\epsilon > \frac{1}{2}$, ϵ-dense STEINER TREE problem is a special case of the network STEINER TREE problem with edge lengths 1 and 2, the problem which is still MAX-SNP-hard, Bern and Plassmann [BP89]. For the dense STEINER TREE problem the existence of a PTAS has been recently proven in Karpinski and Zelikovsky [KZ97b].

Proposition 6. ([KZ97b]) There exists a PTAS for the ϵ-dense STEINER TREE problem.

It is not difficult to see that there is a polynomial time reduction of the ϵ-dense SET COVER to the ϵ-dense STEINER TREE problem, and vice versa. Therefore, the similar result to Proposition 5 holds also for the dense STEINER TREE problem.

Furthermore we conjecture,

Conjecture 2. The dense STEINER TREE problem cannot be computed exactly in polynomial time .

The third problem, VERTEX COVER, is one of the first NP-hard optimization problems for which the approximation algorithms were proposed ([GJ79]). The problem is known to be MAX-SNP-hard, and the well-known 2-approximation algorithm is also believed to be the best possible (cf. [H97]). In Karpinski and Zelikovsky [KZ97b] the new approximation algorithm is designed for dense VERTEX COVER problems beating the approximation ratio 2.

We call a graph $G = (V, E)$ *everywhere* ϵ-*dense* if its *minimum* degree is at least $\epsilon \cdot \mid V \mid$. We call G ϵ-*dense* if $\mid E \mid \geq \epsilon \cdot \mid V \mid^2$.

Proposition 7. ([KZ97b]) There exists a polynomial time approximation algorithm for the VERTEX COVER problem on ϵ-dense graphs with approximation ratio $\frac{2}{2-\sqrt{1-\epsilon}}$.

For the everywhere dense instances we get

Proposition 8. ([KZ97b]) There exists a polynomial time approximation algorithm for the VERTEX COVER problem on everywhere ϵ-dense graphs with approximation ratio $\frac{2}{1+\epsilon}$.

Proposition 7 and 8 show that the *density* do help essentially in approximating the VERTEX COVER problem. Can we expect though existence of a PTAS for the dense VERTEX COVER problem?

The answer is no, as the *everywhere* ϵ-*dense* (and ϵ-*dense*) VERTEX COVER is MAX SNP-hard. (cf. [KZ97b], [CT96]). This is due to the following *densification* construction. Start with a general instance (a graph G with n vertices) of the VERTEX COVER, and densify it by joining its all vertices with all vertices of a clique of size $\frac{\epsilon}{1-\epsilon}n$. The resulting graph is everywhere ϵ-*dense*. An existence of α-approximation algorithm for dense instances of VERTEX COVER entails now also $\alpha(1+\epsilon)$-approximation algorithm for the general VERTEX COVER problem which is MAX-SNP-hard.

Proposition 9. ([CT96], [KZ97b]) The dense VERTEX COVER problem is MAX-SNP-hard.

4 Dense BANDWIDTH MINIMIZATION

We discuss now the problem of approximability of dense instances of the BANDWIDTH problem. The BANDWIDTH problem has a long and very interesting history, and a number of important technical applications (cf., e.g. [CCDG82]). It belongs also to the class of so called layout problems and is one of the hardest in this class ([DSS94]). Its approximability status resembled till recently the BISECTION problem discussed in Section 2 in what there was a general lack of approximation algorithms with essentially sublinear approximation ratios, and on the other hand, till recently, a lack of any innapproximability results. This situation has changed only recently. Blache, Karpinski and Wirtgen [BKW98] proved that there are no PTAS for the BANDWIDTH problem, Unger [U98] extended the hardness result to the arbitrary constants, and Feige [F98] designed the first polylogarithmic factor approximation algorithm for that problem. The situation on dense instances of the BANDWIDTH seemed to be even more difficult than that of the dense BISECTION for which we have constructed a PTAS (see Section 2). For the dense BANDWIDTH however, even the existence of a constant ratio approximation algorithms was an open problem.

We note also that the positive result on existence of a PTAS for the dense BISECTION illustrates the difficulty of proving inapproximability result for the general problem. It indicates that the standard method of reducing *balanced* (50/50) MAX-CUT to BISECTION on the complementary graph cannot work for a good reason. BISECTION still lacks any nontrivial approximation lower bounds, however Feige [F00] was able to prove recently the existence of a polylogarithmic factor approximation algorithm for this problem.

The situation with the BANDWIDTH (the problem which is remotely connected to BISECTION , cf. [DSS94]) is, in fact, even more subtle to analyze in a dense setting. The standard graph operations or a slight densification seem to destroy completely the structure of the instance.

We give now an exact formulation of the problem.

- BANDWIDTH: Given a graph $G = (V, E)$, compute the numbering of its vertices such that the maximum difference between the numbers of adjacent vertices is *minimal*.

We define also the directed BANDWIDTH problem.

- DBANDWIDTH: Given a directed graph $G = (V, E)$, compute the numbering of its vertices as above such that for every vertex v its numbering is greater than any numbering of a vertex u such that $(u, v) \in E$.

The DBANDWIDTH problem corresponds to that of minimizing the *bandwidth* of an upper triangular matrix by simultaneous row and column permutations (cf. Garey, Graham, Johnson and Knuth [GGJK78]).

The problem is known to be NP-hard even if restricted to binary trees (cf. [GGJK78]), or caterpillars with hairs of length at most 3, Monien [M83]. This makes the BANDWIDTH one of the very rare combinatorial problems which are computationally 'hard' for trees. Interestingly, the problem is efficiently computable for complete trees, Smithline [Sm95]. Only a very few special cases of this problem were known before to have sublinear approximation ratio algorithms, among them $\log n$-approximation algorithm for the caterpillars ([HMM91]).

We consider here the BANDWIDTH problem on the everywhere dense graphs. Using a randomized placing technique combined with the special perfect matching design Karpinski, Wirtgen and Zelikovsky [KWZ97] proved

Proposition 10. ([KWZ97]). There exists a randomized polynomial time approximation algorithm for the BANDWIDTH problem on everywhere dense graphs with approximation ratio 3.

Using a more constrained nature of DBANDWIDTH the similar techniques yield.

Proposition 11. ([KWZ97]). There exists a randomized polynomial time approximation algorithm for the DBANDWIDTH problem on everywhere dense graphs with approximation ratio 2.

It is still an open problem whether there are constant ratio approximation algorithms for '*dense*' instances of the BANDWIDTH, and the DBANDWIDTH. A challenging question remains also whether there exist PTASs for the dense BANDWIDTH problems, or whether some of these problem are in fact MAX-SNP-hard.

5 Dense (1,2)-TSP and the Longest Path Problem

The problems left open in [AKK95], and other papers discussed previously were the dense instances of Traveling Salesman and Longest Path problems. The status of these two problems were settled by Fernandez de la Vega and Karpinski [FdVKa98], and Fotakis and Spirakis [FS98].

Let us denote by (1,2)-TSP the *Traveling Salesman Problem with distances 1 and 2*. We call (1,2)-TSP, *d-dense* if the minimum degree of the subgraph spanned by the edges with length 1 is at least dn.

The first explicit lower approximation bounds for general (1,2)-TSP were obtained by Engebretsen [E99] using lower approximation bounds on certain bounded dependency optimization problems obtained recently by Berman and Karpinski [BK99]. The 7/6 upper approximation bound for (1,2)-TSP was proved by Papadimitriou and Yannakakis [PY93].

We define also a *d-dense* Longest Path Problem as a Longest Path Problem restricted to Hamiltonian graphs with the minimum degree dn.

Proposition 12. ([FdVKa98], [FS98]). *d-dense* (1,2)-TSP and *d-dense* Longest Path Problem are both MAX-SNP-hard for every $0 < d < \frac{1}{2}$.

The above results are constructive, and based on Engebretsen [E99] result, they yield the first explicit lower approximation bounds on *d-dense* (1,2)-TSP and *d-dense* Longest Path Problems which depend on the density parameter d of an instance.

6 Summary of Dense Approximation Results

We present here a table summarizing the results of Sections 2-5 with the best known approximation results, and the best up to date nonapproximability results on dense problems.

Problem	**Approx. Ratio**	**Approx. Hardness**	**Ref.**
DENSE MAX-SNP	PTAS	—	[AKK95]
DENSE MAX-CUT	PTAS	—	[AKK95],[FdV96]
DENSE MAX-DCUT	PTAS	—	[AKK95]
DENSE MAX-HYPERCUT(d)	PTAS	—	[AKK95]
DENSE DENSE-K-SUBGRAPH	PTAS	—	[AKK95]
EVERYWHERE DENSE SEPERATOR	PTAS	—	[AKK95]
EVERYWHERE DENSE BISECTION	PTAS	—	[AKK95]
EVERYWHERE DENSE MIN-K-CUT	PTAS	—	[AKK95]
DENSE MIN-LINEAR-ARRANGEMENT	PTAS	—	[AFK96]
DENSE d-DIMENSIONAL-ARRANGEMENT	PTAS	—	[AFK96]
DENSE MIN-CUT-LINEAR-ARRANGEMENT	PTAS	—	[AFK96]
DENSE SET COVER	$\bigcap_c c \cdot \ln n$	OPEN	[KZ97b]
DENSE STEINER TREE	PTAS	—	[KZ97b]
DENSE VERTEX COVER	$\frac{2}{2-\sqrt{1-\varepsilon}}$	MAX-SNP-hard	[KZ97b]
EVERYWHERE DENSE VERTEX COVER	$\frac{2}{1+\varepsilon}$	MAX-SNP-hard	[KZ97b]
EVERYWHERE DENSE BANDWIDTH	3	OPEN	[KWZ97]
EVERYWHERE DENSE DBANDWIDTH	2	OPEN	[KWZ97]
DENSE (1,2)-TSP	7/6	MAX-SNP-hard	[FdVKa98],[FS98]

Table 1: Table of known dense approximability results.

7 Polynomial Time Approximability of Dense Weighted Instances of NP-Hard Problems

The natural instances of optimization problems involve also weights (cf. [GJ79]) while the results studied before were concerned mainly with $0,1$ cases. In Arora, Karger, Karpinski [AKK95], the dense MAX-CUT PTAS can be adjusted as to work also for the dense MAX WEIGHT CUT problem ([GJ79]) for the case of weights being bounded by B. In this case the algorithm produces a cut of weight of at least maximum weight of a cut minus $\epsilon n^2 B$. This and also other bounded weight problems were considered briefly in Goldreich, Goldwasser and Ron [GGR96], and Frieze and Kannan [FK97]. Both papers evaluate the additional costs of handling bounded weights instead of 0,1 weights.

In a recent paper Fernandez de la Vega and Karpinski [FdVK97] gave the first polynomial time approximability characterization of dense (unbounded) weighted instances of MAX WEIGHT CUT, and MAX WEIGHT BISECTION, and some other dense weighted NP-hard optimization problems, in terms of their empirical weight distribution. The crucial point of this paper is a new unbounded weight Sampling Lemma. The reader is referred to [FdVK97] for details.

8 Beyond the Dense Instances of Combinatorial Optimization Problems

We will turn our attention now to the problem which is not dense in our previous sense but can be attacked by using dense methods similar to those developed in [FdVK97]. We define metric MAX-CUT as a MAX-CUT problem with vertices embedded in a metric space. The objective is to divide a set of vertices (points) into two parts as to maximize the sum of the distances between vertices belonging to two different parts.

Fernandez de la Vega and Kenyon [FdVKe98] were able to prove the following result on the metric MAX-CUT problem.

Proposition 13. ([FdVKe98]). Metric MAX-CUT has a PTAS.

Another example of a problem which could be solved by using dense technique of smooth integer programs of [AKK95] was, so called, Maximum Quartet Con-

sistency (MQC) problem of inferring evolutionary trees studied in computational molecular biology, Jiang, Kearney and Li [JKL98].

9 Further Research and Open Problems

It remains to be seen whether the techniques used with success for the dense instances of NP-hard optimization problems, like approximating smooth higher degree integer programs by linear programs, might be useful in approximating general problems. Perhaps some other, different from exhaustive sampling methods can be developed for the nondense instances as well. Another interesting issue is to develop new more efficient techniques for the dense unbounded weight instances of the optimization problems for which costs of allowing weights are not prohibitively high.

On the level of specific dense problems discussed before, it would be interesting to shed some light on the Conjectures 1 and 2. Is there even more dramatic improvement in approximation ratio for the dense SET COVER, like $o(\log n)$, still possible (cf. Proposition 4)?

One of the most challenging open *dense* problems today are the *dense* and *everywhere dense* BANDWIDTH problems. Is there an approximation ratio below 3 (cf. Proposition 10), and even more strongly, is there a PTAS possible for this problem, or perhaps, is this problem 'approximation hard' in some sense?

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