Approximation of the Permanent for Graphs of Density Less than 1/2 is hard

Elias Dahlhaus and Marek Karpinski

University of Bonn

Let $G = (V_1 \cup V_2, E)$ be a bipartite graph with the two partitions V_1 and V_2. Multiple edges are not allowed. We assume that V_1 and V_2 have the same cardinality n. We call G an α-DENSE graph iff each vertex has a degree of at least αn.

We call a probabilistic algorithm an (ϵ, δ)- *approximation algorithm*, iff for given input G, ϵ, δ it computes a number Y, such that the probabiltity that Y is in between $per(G)/(1+\epsilon)$ and $(1+\epsilon)per(G)$ is at least $1-\delta$. Here $per(G)$ is the number of perfect matchings of G (also called the permanent)(see also [4]).

A. Broder[1] stated the following result (see also [4]):

Theorem 1: For 1/2-dense bipartite graphs there is an (ϵ, δ)-approximation algorithm for the permanent, polynomial in n.

Some additional results on approximation algorithms for the permanent are stated in [2].

We can show the following result which is in contrast to theorem 1:

Theorem 2: Given any $\alpha < 1/2$. Then an (ϵ, δ)-approximation algorithm for the permanent of any α-dense bipartite graph, which is polynomial in n, implies the existence of a polynomial time (ϵ, δ)-approximation algorithm for any bipartite graph (approximation completeness).

Proof: The proof is very similar to the proof of matching completeness of the perfect matching problem for α-dense graphs (see [3]. Given any bipartite graph $G = (V_1 \cup V_2, E)$ We consider two copies $C_i = (W_i \cup W_i', E_i')$ of the complete bipartite graph with m vertices per partition. We construct a new bipartite graph G' by joining each vertex of W_i' and each vertex of V_i by an edge. It is easily seen that each perfect matching of G' is the disjoint union of a perfect matching of C_1, a perfect matching of C_2 and a perfect matching of G. Therefore G has k perfect matchings if and only if G' has $2(m!)k$ perfect matchings. Setting the ratio m/n large enough G' is α-dense.

□

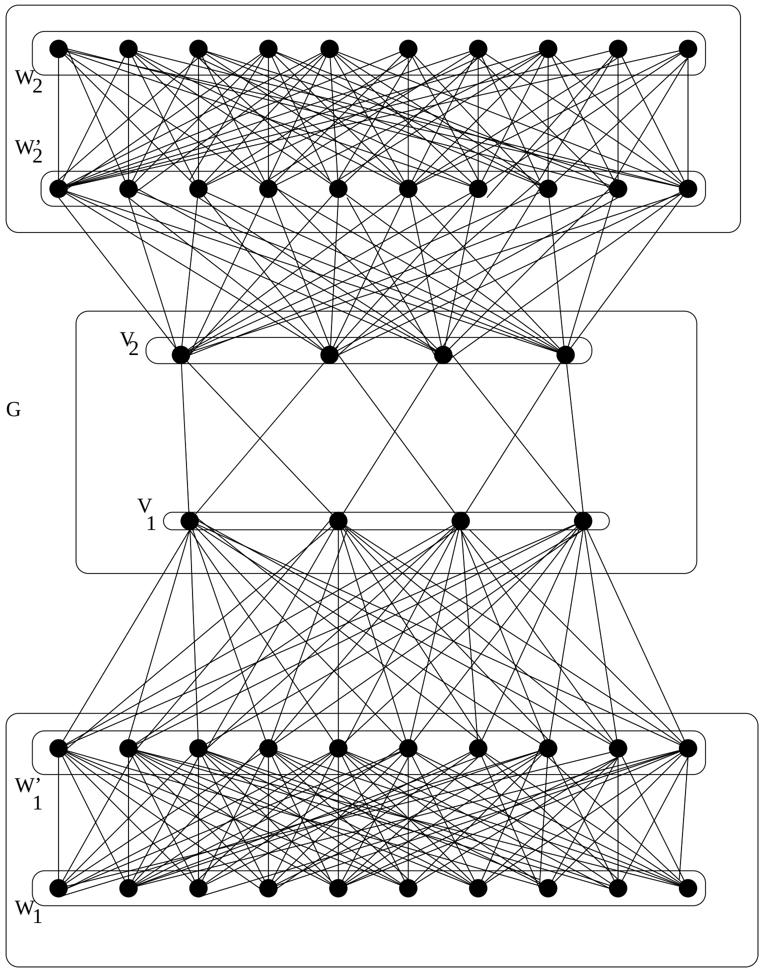

Figure 1: The reduction from any bipartite graph to α-dense graphs

References

[1] A. Broder,*How hard is to marry at random? (On the approximation of the permanent)*, 18-th STOC (1986), pp. 50-58.

[2] P. Dagum, M. Luby, M. Mihail, U. Vazirani, *Polytopes, permanents, and graphs with large factors*, 29-th FOCS(1988), pp. 412-421.

[3] E. Dahlhaus, P. Hajnal, M. Karpinski, *Optimal parallel algorithm for the Hamiltonial cycle problem in dense graphs*, 29-th FOCS, pp. 186-193.

[4] M. Luby, *A survey of approximation algorithms for the permanent*, to appear.