Compact Cellular Algebras and Permutation Groups

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Abstract

Keeping in mind the generalization of Birkhoff's theorem on doubly stochastic matrices we define compact cellular algebras and compact permutation groups. Arising in this connection weakly compact graphs extend compact graphs introduced by G. Tinhofer. It is proved that compact algebras are exactly the centralizer algebras of compact groups. A developed technique enables to get non-trivial examples of compact algebras and groups as well as completely identify compact Frobenius groups and the adjacency algebras of Johnson's and Hamming's schemes. In particular, Petersen's graph proves to be not compact, which answers the question by C. Godsil. A simple polynomial-time isomorphism test for the class of compact cellular algebras (weakly compact graphs) is presented.

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1 Introduction

The starting point of this paper is Birkhoff's theorem on doubly stochastic matrices: each doubly stochastic matrix is a convex combination of permutation matrices. This theorem establishes the following property of the symmetric group: the extreme points of the polytope of doubly stochastic matrices contained in the enveloping algebra of its permutation representation coincide with the permutation matrices corresponding to the elements of the group. We call a permutation group *compact* if it has the property cited above. The characterization of compact groups is the essence of a permutation group approach to the generalization of Birkhoff's theorem. Note that a compact group is necessarily 2-closed, i.e. coincides with the maximal subgroup of the symmetric group having the same 2-orbits.

Another, more combinatorial approach to generalize Birkhoff's theorem was proposed in [11]. Namely, let us associate to an undirected graph Γ the set $\mathrm{DS}(\Gamma)$ of all doubly stochastic matrices commuting with its adjacency matrix. Then Birkhoff's theorem states in fact that for complete graph Γ the following equality holds:

$$\mathrm{DS}(\Gamma) = \mathrm{Conv}(\mathrm{Aut}(\Gamma)), \tag{1}$$

i.e. the polytope $\mathrm{DS}(\Gamma)$ coincides with the convex hull of the permutation matrices corresponding to the automorphisms of Γ. The graphs for which the last equality holds were called Birkhoff's graphs in [13] and *compact* ones in [12]. An important property of compact graphs consists in the fact that their isomorphism can be tested in polynomial time by a variant of refinement procedure.

Comparing the two approaches one can observe that the notion of compact graph is too restrictive. Indeed, there is a simple procedure discovered by B. Weisfeiler and A. Lehman (see [14]) which corresponds to a graph Γ a coherent configuration $C = C(\Gamma)$ such that $\mathrm{Aut}(C) = \mathrm{Aut}(\Gamma)$ (as to coherent configurations see [7]). Moreover, $\mathrm{DS}(C) \subset \mathrm{DS}(\Gamma)$ where $\mathrm{DS}(C)$ is the set of all doubly stochastic matrices centralizing the adjacency algebra of C. Keeping in mind the combinatorial approach one can replace the class of compact graphs by a larger class of *weakly compact* ones, i.e. those for which the condition (1) is satisfied with Γ replaced by C. Note that the isomorphism of weakly compact graphs can still be tested in polynomial time (see section 6). It can be shown that the automorphism group of a weakly compact graph is compact. On the other hand, there exist compact groups (for instance, some regular ones) which can not be obtained in such a way. This suggests to consider the class of compact coherent configurations (compact cellular algebras) as the largest class of compact combinatorial objects.

Let us denote by Mat_V the full matrix algebra over $\mathbf{C}$ on a finite set V, i.e. the set of all complex matrices whose rows and columns are indexed by the elements of V. A subalgebra W of Mat_V is called *cellular* (coherent) if it is closed under the Hadamard (componentwise) multiplication, the Hermitian conjugation and contains the identity matrix I_V and the all one matrix J_V. Each cellular algebra contains a uniquely determined linear base consisting of $\{0,1\}$-matrices summing to J_V, which enables to view it as the adjacency algebra of a coherent configuration.

The automorphism group $\mathrm{Aut}(W)$ of the cellular algebra W consists by definition of all permutations g of V such that the permutation matrix corresponding to g centralizes W. We say that W is *compact* if

$$\mathrm{DS}(W) = \mathrm{Conv}(\mathrm{Aut}(W))$$

where $\mathrm{DS}(W)$ is the polytope of all doubly stochastic matrices centralizing W. As an example of a compact algebra one can take a semiregular one, i.e. the centralizer algebra of semiregular permutation group (theorem 3.3). It turns out (theorem 3.4) that a compact cellular algebra is necessarily Schurian, i.e. coincides with the centralizer algebra of its automorphism group. Thus the two approaches to the generalization of Birkhoff's theorem merge. Namely, there is a 1-1 correspondence

$$W \mapsto \mathrm{Aut}(W), \qquad G \mapsto \mathcal{Z}(G)$$

between compact cellular algebras and compact permutation groups where $\mathcal{Z}(G)$ is the centralizer algebra of the group G.

The main purpose of the paper is to study the properties of compact algebras and groups from both theoretical and algorithmic points of view. We start with studying the structure of a compact cellular algebra by means of combinatorial operations such as fixation, factorization and restriction. In the permutation group language the first of these results states that a setwise stabilizer of a compact group is also compact (theorem 3.7). The second result provides the compactness of the induced action of a compact permutation group on the orbits of each its normal subgroup (theorem 3.8). It is interesting that a transitive constituent of a compact group is not necessarily compact. This follows from the fact that a permutation group having a faithful regular orbit is compact (theorem 3.10).

In section 4 we are interested in the algebraic operations preserving compactness. In particular, we show that the direct and wreath products of cellular algebras are compact if and only if each its operand is compact (theorem 4.1 and theorem 4.2). The case of the tensor product proved to be more complicated. We give an example of two compact cellular algebras the tensor product of which is not compact. However, the tensor product of a compact algebra and a semiregular one is compact (theorem 4.3).

The technique developed in sections 3 and 4 is applied to the characterization of compact objects in various families of cellular algebras and permutation groups (section 5). We prove (theorem 5.1) that each permutation group having a regular Abelian subgroup of index 2 is compact. This generalizes a result from [10] which in our terms means that dihedral groups are compact. We completely describe compact adjacency algebras of Johnson's and Hamming's schemes (theorem 5.3 and theorem 5.5). In particular, it turns out that *Petersen's graph* is not compact (corollary 5.4), which answers a question from [6]. Finally we completely identify compact graphs and algebras of the *Platonic solids* (theorem 5.7) and characterize compact Frobenius groups (theorem 5.8). The last implies that the minimum degree of a non-compact cellular algebra equals 7.

Section 6 is devoted to algorithmic aspects of compact algebras and groups. We present a polynomial-time algorithm which constructs the automorphism group and a canonical labeling of a compact cellular algebra (theorem 6.2). A key point here is the description of the k-orbits of a compact group by means of the Weisfeiler-Lehman algorithm.

Some open problems concerning compact algebras and groups are discussed in section 7.

Notation. As usual by $\mathbf{C}$ and $\mathbf{R}$ we denote the complex field and the real field respectively.

Throughout the paper V denotes a finite set with $n = |V|$ elements. By relations on V me mean subsets of $V \times V$. If E is an equivalence (i.e. reflexive, symmetric and transitive relation) on V, then V/E denotes the set of all equivalence classes modulo E.

The algebra of all complex matrices whose rows and columns are indexed by the elements of V is denoted by Mat_V, its unity (the identity matrix) by I_V and the all one matrix by J_V. For $U \subset V$ the algebra Mat_U is in a natural way identified with a subalgebra of Mat_V.

For $U, U' \subset V$ we denote by $J_{U,U'}$ the {0,1}-matrix with 1's exactly at the places belonging to $U \times U'$.

If $A \in \mathrm{Mat}_V$, then A^T denotes the transpose and A^* the Hermitian conjugate matrix.

If $\varphi : V \to V'$ is a bijection, then A^φ denotes the image of a matrix A with respect to the natural algebra isomorphism from Mat_V to $\mathrm{Mat}_{V'}$ induced by φ.

The group of all permutations of V is denoted by $\mathrm{Sym}(V)$. If $g \in \mathrm{Sym}(V)$, then P_g denotes the permutation matrix corresponding to g. For $S \subset \mathrm{Sym}(V)$ we set

$$P_S = \{P_g : \ g \in S\}$$

and denote by $\mathrm{Conv}(S)$ the convex hull of P_S.

For integers l, m with $l \le m$ by $[l, m]$ we denote the set $\{l, l+1, \ldots, m\}$.

2 Cellular algebras

All undefined below terms concerning permutation groups can be found in [16].

2.1. By a *cellular algebra* on V we mean a subalgebra W of Mat_V for which the following conditions are satisfied:

(C1) $I_V, J_V \in W$;

(C2) $\forall A \in W : \quad A^* \in W$;

(C3) $\forall A, B \in W : \quad A \circ B \in W$,

where $A \circ B$ is the Hadamard (componentwise) product of matrices A and B. It follows from (C2) that W is a semisimple algebra over $\mathbf{C}$.

Each cellular algebra W has a uniquely determined linear basis $\mathcal{R} = \mathcal{R}(W)$ (the standard basis of W) consisting of {0,1}-matrices such that

$$\sum_{R \in \mathcal{R}} R = J_V \quad \text{and} \quad R \in \mathcal{R} \ \Leftrightarrow \ R^* \in \mathcal{R}.$$

Set $\mathrm{Cel}(W) = \{U \subset V : \ I_U \in \mathcal{R}\}$. Each element of $\mathrm{Cel}(W)$ is called a *cell* of W. It is easy to see that

$$V = \bigcup_{U \in \mathrm{Cel}(W)} U \qquad \text{(disjoint union)}.$$

The algebra W is called *homogeneous* if $|\operatorname{Cel}(W)| = 1$.

For $U, U' \in \operatorname{Cel}(W)$ set $\mathcal{R}_{U,U'} = \{R \in \mathcal{R} : \ R \circ J_{U,U'} = R\}$. Then

$$\mathcal{R} = \bigcup_{U,U' \in \operatorname{Cel}(W)} \mathcal{R}_{U,U'} \qquad \text{(disjoint union)}.$$

Moreover, for $R \in \mathcal{R}_{U,U'}$ the number of 1's in the uth row (resp. vth column) of the matrix R does not depend on the choice of $u \in U$ (resp. $v \in U'$). This number is denoted by $d_{out}(R)$ (resp. $d_{in}(R)$).

For each cell $U \in \operatorname{Cel}(W)$ we view the subalgebra $I_U W I_U$ of W as a cellular algebra on U. It is denoted by W_U and called the *homogeneous* component of W corresponding to U. The basis matrices of W_U are in 1-1 correspondence to the matrices of $\mathcal{R}_{U,U}$.

2.2. A large class of cellular algebras comes from permutation groups as follows (see [14]). Let $G \leq \operatorname{Sym}(V)$ be a permutation group. Then its centralizer algebra $\mathcal{Z}(G) \subset \operatorname{Mat}_V$ is a cellular algebra on V. Its standard basis consists of the adjacency matrices of the 2-orbits of G. In particular, the cells of $\mathcal{Z}(G)$ coincide with the orbits of G.

A cellular algebra W is called *semiregular* if $d_{in}(R) = d_{out}(R) = 1$ for all $R \in \mathcal{R}(W)$. A homogeneous semiregular algebra is called *regular*. It is easy to see that semiregular (regular) algebras coincide with centralizer algebras of semiregular (regular) permutation groups.

Two cellular algebras W and W' on V and V' are called *isomorphic* if $W^{\varphi} = W'$ (as sets) for some bijection $\varphi : V \to V'$ called an *isomorphism* from W to W'. Clearly, φ induces a bijection between the sets $\mathcal{R}(W)$ and $\mathcal{R}(W')$. The group of all isomorphisms from W to itself contains a normal subgroup

$$\operatorname{Aut}(W) = \{\varphi \in \operatorname{Sym}(V) |\ A^{\varphi} = A,\ A \in W\}$$

called the *automorphism group* of W. If $W = \mathcal{Z}(\operatorname{Aut}(W))$, then W is called *Schurian*. It follows from [16] that there exist cellular algebras which are not Schurian (see also [3]).

2.3. Let E be an *equivalence* on V. Set $I_E = \sum_{U \in V/E} J_U/|U|$. It is easy to see that the linear map

$$i_E : \operatorname{Mat}_{V/E} \to \operatorname{Mat}_V, \quad e_{U,U'} \mapsto \frac{1}{\sqrt{|U||U'|}} J_{U,U'} \tag{2}$$

where $e_{U,U'}$ is a matrix unit of $\operatorname{Mat}_{V/E}$, is an injective ring homomorphism preserving the orthogonality with respect to the Hadamard multiplication, $i_E(I_{V/E}) = I_E$ and $\operatorname{Im}(i_E) = I_E \operatorname{Mat}_V I_E$.

Let now W be a cellular algebra on V and E be an equivalence of W, i.e. $I_E \in W$. Set

$$W/E = i_E^{-1}(I_E W I_E).$$

Then W/E is a subalgebra of $\operatorname{Mat}_{V/E}$ isomorphic to $I_E W I_E$ as a matrix algebra. It can be proved that W/E is really a cellular algebra on V/E called the *factoralgebra* of the algebra W modulo E. Moreover, its basis matrices are multiples of the matrices $i_E^{-1}(I_E R I_E)$ with $R \in \mathcal{R}(W)$.

Since the matrix I_E centralizes $P_{\mathrm{Aut}(W)}$, each $g \in \mathrm{Aut}(W)$ induces a permutation of the set V/E which clearly belongs to $\mathrm{Aut}(W/E)$. This defines a group homomorphism

$$\varphi_E : \mathrm{Aut}(W) \to \mathrm{Aut}(W/E), \tag{3}$$

the kernel of which coincides with the subgroup of $\mathrm{Aut}(W)$ leaving each class of E fixed.

An equivalence E of W is called *central* if $I_E A = A I_E$ for all $A \in W$. It is easy to see that in this case each class of E is contained in a cell of W. The factoralgebras modulo equivalences satisfying the last condition were introduced and studied in [14].

2.4. The set of all cellular algebras on V is put in order by inclusion. The algebra Mat_V is obviously the greatest element of the set. The least one is the simplex $S(V)$, i.e. a cellular algebra with two basis matrices I_V and $J_V - I_V$. We write $W \leq W'$ if W is a subalgebra of W'.

Given $X \subset \mathrm{Mat}_V$, the cellular closure of X, i.e. the smallest cellular algebra containing X, is denoted by $[X]$. If W is a cellular algebra on V, then $W[X]$ denotes $[W \cup X]$. We use notation W_v and W_E if $X = \{I_{\{v\}}\}$, $v \in V$, and $X = \{I_U\}_{U \in V/E}$ where E is an equivalence on V respectively. Finally, we write $[A]$ and $W[A]$ instead of $[\{A\}]$ and $W[\{A\}]$.

3 Compactness

3.1. For an arbitrary set $X \subset \mathrm{Mat}_V$ let us denote by $\mathrm{DS}(X)$ the set of all doubly stochastic matrices of Mat_V commuting with each matrix of X.

Definition 3.1 *The set X is called compact if all extreme points of $\mathrm{DS}(X)$ are integral. We say that X is weakly compact if the set $[X]$ is compact.*

Note that the set of all integral extreme points of $\mathrm{DS}(X)$ coincides with the set $\{P_g : g \in \mathrm{Aut}(X)\}$ where $\mathrm{Aut}(X) = \{g \in \mathrm{Sym}(V) : A^g = A,\ A \in X\}$. So the compactness of X means that there are no other extreme points. Certainly, if X is a cellular algebra, the compactness of X is equivalent to its weak compactness.

Proposition 3.2 *A compact set is weakly compact.*

Proof. Let $X \subset \mathrm{Mat}_V$ be a compact set and $W = [X]$ be its cellular closure. Then, clearly $\mathrm{Aut}(X) = \mathrm{Aut}(W)$, $\mathrm{DS}(X) \supset \mathrm{DS}(W)$. So $\mathrm{DS}(W) \subset \mathrm{Conv}(\mathrm{Aut}(W))$ due to the compactness of X. On the other hand, since $P_{\mathrm{Aut}(W)} \subset \mathrm{DS}(W)$, we have $\mathrm{DS}(W) \supset \mathrm{Conv}(\mathrm{Aut}(W))$, which completes the proof.∎

An undirected graph Γ is called compact (weakly compact), if the set $\{A_\Gamma\}$ is compact (weakly compact) where A_Γ is the adjacency matrix of Γ. Note that this definition of compact graph coincides with that of [12]. Proposition 3.2 shows that a compact graph is weakly compact. The converse statement is not true. A counterexample is given by any regular graph Γ for which $[A_\Gamma] = \mathrm{Mat}_V$ where V is the vertex set of Γ. Nevertheless in a number of cases the weak compactness of a graph Γ implies its compactness. This happens, for instance, if $[A_\Gamma]$ coincides with the matrix algebra generated by A_Γ. The last condition is clearly satisfied for a connected distance-regular graph Γ.

3.2. It follows from the definition that the study of weak compactness is reduced to the study of compact cellular algebras. The simplest example of a compact cellular algebra is Mat_V. It follows from Birkhoff's theorem on doubly stochastic matrices that the symplex $S(V)$ is also compact.

Theorem 3.3 *A semiregular (in particular, regular) cellular algebra is compact.*

Proof. The automorphism group of semiregular algebra W is a semiregular permutation group and the corresponding permutation matrices are pairwise orthogonal with respect to the Hadamard multiplication. Since any matrix commuting with all matrices of W is a linear combination of these permutation matrices, the compactness of W follows.■

In [5] it was proved that a compact distance-regular graph is distance-transitive. In our terms this means that the cellular algebra of a compact distance-regular graph is Schurian. We generalize this statement as follows.

Theorem 3.4 *A compact cellular algebra is Schurian.*

Proof. Let W be a cellular algebra on V. Set

$$\mathrm{C}(W) = \{A \in \mathrm{Mat}_V : \ AB = BA, \ B \in W\}.$$

According to [15] the algebra $\mathrm{C}(W)$ is semisimple and $\mathrm{C}(\mathrm{C}(W)) = W$.

Lemma 3.5 $\mathrm{C}(W)$ *is spanned by the set* $\mathrm{DS}(W)$.

Proof. Let A be a real matrix belonging to $\mathrm{C}(W)$. Then $AI_U = I_U A$ for all $U \in \mathrm{Cel}(W)$, whence it follows that A is a block-diagonal matrix with blocks corresponding to the cells of W. Moreover, since A commutes with all matrices $J_{U,U'}$, $U, U' \in \mathrm{Cel}(W)$, we see that the row (column) sums of the matrix A coincide. So there exist $\alpha, \beta \in \mathbf{R}$ such that $\alpha A + \beta J_W \in \mathrm{DS}(W)$. Since the algebra W is defined over $\mathbf{R}$, so is $\mathrm{C}(W)$, which completes the proof.■

Let now W be a compact cellular algebra. Then by lemma 3.5

$$\mathrm{C}(W) = \mathrm{Env}(\mathrm{Aut}(W))$$

where in the right side the enveloping algebra of permutation group $\mathrm{Aut}(W)$ stands. Taking into account that W is a semisimple algebra over $\mathbf{C}$, we get

$$W = \mathrm{C}(\mathrm{C}(W)) = \mathrm{C}(\mathrm{Env}(\mathrm{Aut}(W))) = \mathcal{Z}(\mathrm{Aut}(W)).$$

This proves that W is Schurian.■

The last theorem shows that the combinatorial approach to Birkhoff's theorem gives no compact combinatorial objects different from centralizer algebras of compact groups.

Definition 3.6 *A permutation group G is called compact if each doubly stochastic matrix contained in* $\mathrm{Env}(G)$ *is a convex combination of P_g, $g \in G$.*

It should be noted that the class of compact groups can be viewed as the largest class of permutation groups for which the analog of Birkhoff's theorem on doubly stochastic matrices is valid. It easily follows from the definition that each compact group is necessarily 2-closed, i.e. coincides with the automorphism group of some subset of Mat_V.

Theorem 3.4 shows that there is a 1-1 correspondence ($W \mapsto \mathrm{Aut}(W)$, $G \mapsto \mathcal{Z}(G)$) between compact cellular algebras and compact permutation groups on V. So all the results of the paper can be formulated both for compact groups and compact algebras. In each case we choose the way more suitable for us.

3.3. As it is shown in section 5 there are examples of non-compact cellular algebras for all $n \geq 7$. Each of them is an overalgebra of a simplex. So in general an overalgebra of a compact algebra is not necessarily compact. However, there is a simple way to construct compact overalgebras.

Theorem 3.7 *Let $W \leq \mathrm{Mat}_V$ be a compact cellular algebra. Then for each equivalence E on V the algebra W_E is also compact. In particular, the algebra $W[I_U]$ is compact for all $U \subset V$.*

Proof. Let $A \in \mathrm{DS}(W_E)$. Then $A \in \mathrm{DS}(W)$ and so by the compactness of W we have

$$A = \sum_{g\in\mathrm{Aut}(W)} \lambda_g P_g, \qquad \sum_g \lambda_g = 1, \quad \lambda_g \geq 0.$$

Since A is a block-diagonal matrix whose blocks coincide with the classes of E, the inequality $\lambda_g > 0$ implies that g leaves fixed each class of E, i.e. $g \in \mathrm{Aut}(W_E)$. Thus $A \in \mathrm{Conv}(\mathrm{Aut}(W_E))$.$\blacksquare$

3.4. It is easy to see that the factor of a regular cellular algebra modulo its central equivalence is also regular and hence a compact one. This observation can be generalized to all compact cellular algebras as follows.

Theorem 3.8 *Let $W \leq \mathrm{Mat}_V$ be a compact cellular algebra. Then for each central equivalence E of W the factoralgebra W/E is also compact. Moreover, a natural group homomorphism $\varphi_E : \mathrm{Aut}(W) \to \mathrm{Aut}(W/E)$ is a surjection.*

Proof. Let $A \in \mathrm{DS}(W/E)$. Then $i_E(A) \in \mathrm{DS}(W)$ where i_E is the injection (2). Indeed, since E is a central equivalence of W, each its class is contained in some cell of W. Besides, A is a block-diagonal matrix with blocks corresponding to the cells of W/E. So the matrix $i_E(A)$ is also block-diagonal with blocks corresponding to the cells of W, whence it follows that it is doubly stochastic. Finally, since $i_E(W/E) = I_E W I_E$ and $i_E(A)I_E = I_E i_E(A) = i_E(A)$, for any $B \in W$ we have

$$\begin{aligned} i_E(A)B = i_E(A)I_E B = i_E(A)i_E(\bar{B}) &= i_E(A\bar{B}) = \\ = i_E(\bar{B}A) &= i_E(\bar{B})i_E(A) = BI_E i_E(A) = BI_E(A), \end{aligned}$$

where $\bar{B} = i_E^{-1}(I_E B) \in W/E$.

It follows from the compactness of W that

$$i_E(A) = \sum_{g\in\mathrm{Aut}(W)} \lambda_g P_g, \qquad \sum_g \lambda_g = 1, \quad \lambda_g \geq 0.$$

Multiplying the both sides by I_E and using the definitions of i_E and φ_E (see (3)) we get

$$i_E(A) = \sum_g \lambda_g (P_g I_E) = \sum_g \lambda_g i_E(P_{\varphi_E(g)}).$$

So by the injectivity of i_E the matrix A belongs to $\mathrm{Conv}(\varphi_E(\mathrm{Aut}(W)))$. This proves the both statements of the theorem (the second one by setting $A = P_h$, $h \in \mathrm{Aut}(W/E)$).$\blacksquare$

Remark 3.9 *The statement of the theorem is no longer true if E is not central. Indeed, each homogeneous Schurian algebra W is isomorphic to a factoralgebra of the centralizer algebra of the regular representation of* $\mathrm{Aut}(W)$. *However, as we will see in section 5, there exist homogeneous Schurian algebras which are not compact.*

It follows from theorems 3.7 and 3.8 that if W is a compact cellular algebra and E is its central equivalence, then the algebras W_E and W/E are also compact. As the example of the icosahedron shows (see proof of theorem 5.7), the converse statement is not true.

3.5. The restriction of a cellular algebra to a cell can be viewed as some kind of factorization. The following statement shows that a homogeneous component of a compact cellular algebra is not necessarily compact (see also remark 3.9).

Theorem 3.10 *A permutation group having a faithful regular orbit is compact.*

Proof. Let $G \leq \mathrm{Sym}(V)$ be a permutation group satisfying the hypothesis of the theorem and W be its centralizer algebra. Then W has a cell U_0 such that the algebra W_{U_0} is regular and $G \cong \mathrm{Aut}(W_{U_0})$. So $d_{out}(R) = 1$ for all $R \in \mathcal{R}_{U_0,U}$, $U \in \mathrm{Cel}(W)$.

Let $A \in \mathrm{DS}(W)$. Then by the compactness of W_{U_0}

$$A \circ J_{U_0} = \sum_{g' \in \mathrm{Aut}(W_{U_0})} \lambda_{g'} P_{g'}, \qquad \sum_{g'} \lambda_{g'} = 1, \quad \lambda_{g'} \geq 0. \tag{4}$$

Let us prove that

$$A = \sum_{g \in G} \lambda_{\varphi(g)} P_g, \tag{5}$$

where $\varphi : G \to \mathrm{Aut}(W_{U_0})$ is the restriction isomorphism. Denote by A' the difference between the left and the right sides of (5). Then $A' \in \mathrm{C}(W)$ and $A' \circ J_{U_0} = 0$ by (4). So for each $U \in \mathrm{Cel}(W)$ and $R \in \mathcal{R}_{U_0,U}$ we have

$$RA'_U = RA' = A'R = A'_{U_0}R = 0$$

where $A'_U = A' \circ J_U$. Since $d_{out}(R) = 1$, it follows that $A'_U = 0$ for all U, i.e. $A' = 0$. This proves (5).■

4 Operations preserving compactness

4.1. Let $W_1 \leq \mathrm{Mat}_{V_1}$ and $W_2 \leq \mathrm{Mat}_{V_2}$ be cellular algebras. Following [14] let us define their direct sum being a cellular algebra on the disjoint union of V_1 and V_2, by

$$W_1 \boxplus W_2 = [\mathcal{R}(W_1) \cup \mathcal{R}(W_2)].$$

It is easy to see that $\mathrm{Aut}(W_1 \boxplus W_2)$ is isomorphic to $\mathrm{Aut}(W_1) \times \mathrm{Aut}(W_2)$.

Theorem 4.1 *The cellular algebra $W_1 \boxplus W_2$ is compact iff so are W_1 and W_2.*

Proof. It immediately follows from the definition that

$$\mathrm{DS}(W_1 \boxplus W_2) = \mathrm{DS}(W_1) + \mathrm{DS}(W_2), \qquad P_{\mathrm{Aut}(W_1 \boxplus W_2)} = P_{\mathrm{Aut}(W_1)} + P_{\mathrm{Aut}(W_2)}$$

(as sets) and the sums are direct. So the theorem follows. ∎

Certainly, the definition of the direct sum and the theorem can be extended to an arbitrary number of summands.

4.2. Let us define the wreath product of cellular algebras $W_1 \le \mathrm{Mat}_{V_1}$, $W_2 \le \mathrm{Mat}_{V_2}$ being a cellular algebra on the set $V_1 \times V_2$, by

$$W_1 \wr W_2 = [\mathcal{R}(W_1) \otimes I_{V_2} \cup J_{V_1} \otimes \mathcal{R}(W_2)]$$

where $\otimes$ denotes the Kronecker product of matrices. It can be verified that $\mathrm{Aut}(W_1 \wr W_2)$ is isomorphic to the wreath product of $\mathrm{Aut}(W_1)$ and $\mathrm{Aut}(W_2)$. For homogeneous W_1 and W_2 our definition is compatible with that of [14].

Let us denote by E the equivalence on $V_1 \times V_2$ defined by the coincidence of the second coordinates. Then E is an equivalence of $W_1 \wr W_2$, the restriction of $W_1 \wr W_2$ to any class of E is isomorphic to W_1 and

$$(W_1 \wr W_2)/E = W_2, \qquad (W_1 \wr W_2)_E = \underbrace{W_1 \boxplus \cdots \boxplus W_1}_{V_2 \text{ times}}. \tag{6}$$

Theorem 4.2 *The cellular algebra $W_1 \wr W_2$ is compact iff so are W_1 and W_2.*

Proof. Let algebras W_1 and W_2 be compact and $A \in \mathrm{DS}(W)$ where $W = W_1 \wr W_2$. We view A as a block matrix each block $A_{u,v}$, $u, v \in V_2$, of which is a matrix of Mat_{V_1}. Since A commutes with E, we see that the row (column) sums of the matrix $A_{u,v}$ coincide. Let us denote this number by $a_{u,v}$ and consider the matrix $\bar{A} = (a_{u,v})_{u,v \in V_2}$. Clearly, $\bar{A}$ is a doubly stochastic matrix of Mat_{V_2}. The condition of commuting A with each matrix $J_{V_1} \otimes R$, $R \in \mathcal{R}(W_2)$ shows that the matrix $\bar{A}$ centralizes the algebra W_2, i.e. $\bar{A} \in \mathrm{DS}(W_2)$. It follows from the compactness of W_2 that

$$\bar{A} = \sum_{g \in \mathrm{Aut}(W_2)} \lambda_g P_g, \qquad \sum_g \lambda_g = 1, \quad \lambda_g \ge 0. \tag{7}$$

Let us define a block matrix $\widetilde{A} = (\widetilde{A}_{u,v})_{u,v \in V_2}$ belonging to $\mathrm{Mat}_{V_1 \times V_2}$ by

$$\widetilde{A}_{u,v} = \begin{cases} a_{u,v}^{-1}, & \text{if } a_{u,v} \ne 0; \\ 0, & \text{otherwise.} \end{cases}$$

It follows from the definition that either $\widetilde{A}_{u,v} = 0$ or $\widetilde{A}_{u,v}$ is a doubly stochastic matrix of Mat_{V_1}. By a straightforward checking we get from (7) that

$$A = \sum_{g \in \mathrm{Aut}(W_2)} \lambda_g A_g \tag{8}$$

where $A_g = \widetilde{A} \circ (J_{V_1} \otimes P_g)$. Since each block of the matrix $\widetilde{A}$ commutes with all matrices of W_1, each nonzero block of A_g belongs to $\mathrm{DS}(W_1)$, and so it is a convex combination of the permutation matrices corresponding to the automorphisms of W_1. So

$$A_g = \sum_{\bar{h} \in G_1^{V_2}} \mu_{\bar{h}} P_{(\bar{h};g)}, \qquad \sum_{\bar{h}} \mu_{\bar{h}} = 1, \quad \mu_{\bar{h}} \ge 0 \tag{9}$$

where $G_1 = \mathrm{Aut}(W_1)$ and $(\bar{h}; g) = (\{h_v\}_{v \in V_2}; g)$ is the permutation of $V_1 \times V_2$ given by

$$(u, v)^{(\bar{h};g)} = (u^{h_v}, v^g).$$

Since the permutation $(\bar{h}; g)$ belongs to $G_1 \wr \mathrm{Aut}(W_2) = \mathrm{Aut}(W)$, it follows from (8) and (9) that $A \in \mathrm{Conv}(\mathrm{Aut}(W))$.

Conversely, let W be a compact cellular algebra. Then by theorem 3.7, the second equality of (6) and theorem 4.1 the algebra W_1 is compact. Let us denote by E_0 the coarsest central equivalence of W containing in E. Then

$$W/E_0 = \mathrm{Mat}_{\mathrm{Cel}(W_1)} \wr W_2 = \mathrm{Mat}_{\mathrm{Cel}(W_1)} \otimes W_2.$$

It is easy to see that for any finite set S

$$\mathrm{DS}(\mathrm{Mat}_S \otimes W_2) = I_S \otimes \mathrm{DS}(W_2), \quad P_{\mathrm{Aut}(\mathrm{Mat}_S \otimes W_2)} = I_S \otimes P_{\mathrm{Aut}(W_2)}. \tag{10}$$

Thus the compactness of W_2 follows from the compactness of W/E_0 (see theorem 3.8).∎

4.3. Following [14] let us define the tensor product of cellular algebras $W_1 \leq \mathrm{Mat}_{V_1}$ and $W_2 \leq \mathrm{Mat}_{V_2}$ being a cellular algebra on $V_1 \times V_2$, by

$$W_1 \otimes W_2 = [\mathcal{R}(W_1) \otimes \mathcal{R}(W_2)].$$

It can be verified that $\mathrm{Aut}(W_1 \otimes W_2)$ is isomorphic to $\mathrm{Aut}(W_1) \times \mathrm{Aut}(W_2)$. It should be mentioned that the cellular algebra $W_1 \otimes W_2$ as a matrix algebra coincides with the tensor product of the matrix algebras W_1 and W_2. It immediately follows from the definition that $W_1 \wr W_2 \leq W_1 \otimes W_2$.

The tensor product of compact cellular algebras is not necessarily compact. For example we by using Fukuda's program (see [4]) found 1116 extreme points of the polytope $\mathrm{DS}(S(3) \otimes S(3))$ whereas $|\mathrm{Aut}(S(3) \otimes S(3))| = 36$. However the following statement holds.

Theorem 4.3 *Let W_2 be a semiregular cellular algebra. Then the algebra $W_1 \otimes W_2$ is compact iff so is W_1.*

Proof. Since each semiregular algebra is isomorphic to the tensor product of a regular algebra and a full matrix algebra, we can assume that the algebra W_2 is regular (see (10)).

Let W_1 be a compact cellular algebra and $A \in \mathrm{DS}(W)$ where $W = W_1 \otimes W_2$. Then

$$A = \sum_{u,v \in V_2} A_{u,v} \otimes e_{u,v}, \qquad A_{u,v} \in \mathrm{Mat}_{V_1}$$

where $e_{u,v}$ is the matrix unit of Mat_{V_2} corresponding to (u, v). Since A commutes with all matrices $I_{V_1} \otimes R$, $R \in \mathcal{R}(W_2)$, we have

$$A_{u,v} = A_{u^h, v^h}, \quad h \in G$$

where $G = \{h \in \mathrm{Sym}(V_2) : \ P_h \in \mathcal{R}(W_2)\}$. So

$$A = \sum_{g \in \mathrm{Aut}(W_2)} A_g \otimes P_g, \qquad A_g \in \mathrm{Mat}_{V_1},$$

since $\{P_g : g \in \mathrm{Aut}(W_2)\} = \{\sum_{h\in G} e_{u^h,v^h} : u,v \in V_2\}$ due to the regularity of W_2. By using the fact that A commutes with all matrices from $W_1 \otimes I_{V_2}$ we conclude that $A_g \in \mathrm{DS}(W_1)$ for all g. By the compactness of W_1 we see that $A_g \in \mathrm{Conv}(\mathrm{Aut}(W_1))$. Therefore $A \in \mathrm{Conv}(\mathrm{Aut}(W))$.

Conversely, suppose W to be a compact cellular algebra. Let us denote by E the equivalence on $V_1 \times V_2$ defined by the coincidence of the first coordinates. Since W_2 is homogeneous, E is a central equivalence of W. Thus the compactness of W_1 follows from theorem 3.8 and the equality $W/E = W_1$.■

5 Examples

5.1. We start with describing an infinite family of compact cellular algebras and permutation groups. The following statement generalizes the result proved in [10].

Theorem 5.1 *Let G be a permutation group having an Abelian regular subgroup H of index 2. Then G is compact.*

Proof. Let $G \le \mathrm{Sym}(V)$ and H be the corresponding regular subgroup. Clearly, H is a normal subgroup of G and $G = H \cup tH$ where $t \in G_v$ for some $v \in V$, $t^2 = 1$. Set

$$T = \bigcup_{v\in V} G_v, \qquad S = tT.$$

According to [16, Ch. 4] identify V with H so that v^h corresponds to $h \in H$. Here G is identified with a subgroup of $\mathrm{Sym}(H)$ so that

$$h^g = \begin{cases} hg, & \text{if } g \in H; \\ (tht)h_1, & \text{if } g = th_1,\ h_1 \in H. \end{cases}$$

In particular, $h^t = tht$ for all $h \in H$.

It follows from the definitions that in the above notation $h \in S$ if and only if there exists $k \in H$ for which $h = k \cdot (k^t)^{-1}$. Thus the commutativity of H implies that S is a subgroup of H. Let us prove that

$$\sum_{g\in S} P_g = \sum_{g\in T} P_g. \tag{11}$$

For $h \in H$ we have $(h, h^t) = (h, h^{h'})$ where $h' = k(k^t)^{-1} \in S$ with $k = h^{-1}$. Thus $P_t \circ \sum_{g\in S} P_g = P_t$. Since S is a subgroup of H, the same is true with t replaced by any element of T. So the matrix in the right side of (11) is not changed multiplied by the matrix in the left side. Since $|S| = |T|$, the equality (11) follows.

Let A be a doubly stochastic matrix belonging to $\mathrm{Env}(G)$. Then

$$A = \sum_{g\in G} \lambda_g P_g, \qquad \lambda_g \in \mathbf{R}. \tag{12}$$

Without loss of generality assume that the number of negative λ_g in (12) is minimal possible. Let $\lambda_{g_0} < 0$ for some $g_0 \in G$. We can also assume that $g_0 = 1$, since otherwise A can be replaced by $P_{g^{-1}}A$. Then

$$A = \sum_{g\in G\setminus(S\cup T)} \lambda_g P_g + \sum_{g\in S}(\lambda_g + |\lambda_1|)P_g + \sum_{g\in T}(\lambda_g - |\lambda_1|)P_g.$$

Since $P_g \circ P_1 \neq 0$ for all $g \in T$, we see that $\lambda_g - |\lambda_1| = \lambda_g + \lambda_1 \geq 0$. On the other hand, the coefficient at 1 of the last decomposition equals 0. So the number of negative coefficients of this decomposition is less that of decomposition (12). This contradicts to the choice of the latter.■

Remark 5.2 *The condition of the commutativity of H is essential. Indeed, let G be the permutation group arising from the action of* $\mathrm{Sym}(4)$ *on the right cosets of the subgroup generated by a transposition. Then G has a regular subgroup H of index 2 isomorphic to* $\mathrm{Alt}(4)$. *This permutation group is not compact: by using Fukuda's program [4] we found 162 extreme points of the corresponding polytope whereas only 24 of them are integral.*

5.2. In paper [6] it was shown that the triangle graphs T_n are compact for $n \leq 4$ and are not compact for all $n \geq 6$. The compactness of T_5, the complement to Petersen's graph, was an open problem. Nothing was known about the compactness of the Johnson graph $J_{n,k}$ for $k \geq 3$.

Let us denote by $J(n,k)$, $1 \leq k \leq n/2$, the adjacency algebra of the Johnson scheme with parameters n and k, i.e. the centralizer algebra of the action of $\mathrm{Sym}(n)$ on the set of all k-elements subsets of $[1,n]$. It is known (see [2]) that $J(n,k)$ is a commutative homogeneous cellular algebra of dimension $k+1$. The adjacency matrix of the Johnson graph $J_{n,k}$ belongs to the standard basis of the algebra $J(n,k)$ and generates it as a matrix algebra.

Theorem 5.3 *The cellular algebra $J(n,k)$ is compact iff $k = 1$ or $(n,k) = (4,2)$.*

Proof. Since $J(n,1) = S(n)$ and $S(n)$ is compact (see section 3), we assume $k \geq 2$. Besides, $J(4,2) \cong S(2) \wr S(3)$. By theorem 4.2 the last algebra is compact. Thus we assume in addition that $n \geq 5$.

Let us denote by R the adjacency relation of the Johnson graph $J_{n,k}$:

$$R = \{(S,S') : \ S,S' \subset [1,n], \ |S| = |S'| = k, \ |S \cap S'| = k-1\}. \tag{13}$$

and by A its adjacency matrix. Then A is a basis matrix of the algebra $J(n,k)$ (see above). Let us prove the following statement:

$$\forall g \in \mathrm{Sym}(n) \ \exists S \subset [1,n] : \ |S| = k, \ (S, S^g) \notin R. \tag{14}$$

Let $g \in \mathrm{Sym}(n)$ and $F = \{v \in [1,n] : \ v^g = v\}$ be the set of all fixed points of g. If $|F| \geq k$, then any k-subset S of F satisfies (14). Otherwise, since $n \geq 5$, there exist two distinct points u, v such that $\{u^g, v^g\} \cap \{u, v\} = \emptyset$. Since $k \geq 2$, there exists a k-subset S of $[1,n]$ for which $u, v \in S$ and $u^g, v^g \notin S$. Then $|S \cap S^g| \leq k-2$ which implies that $(S, S^g) \notin R$ and proves (14).

Due to the commutativity of the algebra $W = J(n,k)$, the matrix $\frac{1}{d}A$ belongs to $\mathrm{DS}(W)$ where d is the degree of R. To prove the non-compactness of W it suffices to check that

$$\forall g \in G \ \exists u \in V : \quad A_{u,u^g} = 0 \tag{15}$$

where $G = \mathrm{Aut}(W)$ and $A = (A_{u,v})$. If $n \geq 2k+1$, then $G = \mathrm{Sym}(n)$ (see Lemma 2.1.3 in Appendix 2 of [2]) and (15) follows from (14). If $n = 2k$, then $G = \mathrm{Sym}(n) \times \{1, t\}$ where t is the permutation moving a k-subset of $[1,n]$ to its complement. For $g \in \mathrm{Sym}(n)$ we reason as above. If $g = ht$ with $h \in \mathrm{Sym}(n)$, then the inequality (15) for it is the consequence of the analog of statement (14) with S^g replaced by its complement (in the proof we have to choose a k-subset S containing u, v, u^g, v^g). This completes the proof of the theorem.■

It follows from the theorem that the triangle graph $T_n = J_{n,2}$ is not compact for $n \geq 5$. Since Petersen's graph is the complement to T_5, we get the following statement.

Corollary 5.4 *Petersen's graph is not compact.*

5.3. For a positive integer n let us denote by $H(n)$ the centralizer algebra of the permutation group $G^{(n)}$ which is the wreath product of Sym(2) and Sym(n) acting on the set $\{0,1\}^n$. Clearly, $H(n)$ is a homogeneous algebra. According to [3] $\mathrm{Aut}(H(n)) = G^{(n)}$ for all n. Certainly, $H(n)$ coincides with the adjacency algebra of the Hamming scheme with parameters $n, 2$ (see [2]). Notice that $H(n)$ is generated as a matrix algebra by the adjacency matrix of the n-dimensional cube.

Theorem 5.5 *The cellular algebra $H(n)$ is compact iff $n \leq 3$.*

Proof. It is easy to see that the algebras $H(1), H(2)$ and $H(3)$ are isomorphic to $S(2)$, $S(2) \wr S(3)$ and $S(4) \otimes S(2)$ respectively. So the compactness of them follows from theorem 4.2 and theorem 4.3 and the compactness of the simplex. Let us consider the algebra $W = H(4)$. It is easy to see that $W/E \cong S(4) \wr S(2)$ where E is the equivalence of W with classes of cardinality 2. Then due to theorem 3.8 the algebra W is not compact, since $|\mathrm{Aut}(W)| = 2^4 4! = 384$ and $\mathrm{Aut}(W/E) = (4!)^2 2 = 1152$.

Let $n \geq 5$. Set $W = H(n)$ and W' to be the centralizer algebra of the action of Sym(n) on the set $2^{[1,n]}$ (naturally bijective to $\{0,1\}^n$) of all subsets of $[1,n]$. Below we will show that W' is a non-compact cellular algebra. It will imply that so is W, since otherwise by theorem 3.7 $W' \cong W_v$, $v \in \{0,1\}^n$, and the algebra W' would be compact.

Let us consider the algebra W'. Index the cells of W' by the numbers $0, 1, \ldots, n$ so that the kth cell consists of all k-subsets of $[1,n]$. Set

$$A = \sum_g P_g - \lambda I_{2^{[1,n]}}$$

where g runs over all elements of $\mathrm{Aut}(W')$ corresponding to the transpositions of Sym(n) and $\lambda = \binom{n}{2} - k_0(n - k_0)$ with $k_0 = \lfloor \frac{n}{2} \rfloor$. Note that the elements of the matrix A are non-negative integers, since the Hadamard product of A and the unity of the kth homogeneous component of W' is a multiple of this unity with the coefficient $k_0(n - k_0) - k(n-k)$ which is not negative for all k. Since $|S \cap S^g| \geq |S| - 1$ for any set $S \subset [1,n]$ and a transposition $g \in \mathrm{Sym}(n)$, the restriction of A to the k_0th cell of W' is a multiple of the adjacency matrix of the relation R defined in (13) (with $k = k_0$). As we saw in proving theorem 5.3 no multiple of this matrix belongs to $\mathrm{DS}(W_0)$ where W_0 is the k_0th homogeneous component of W' (coinciding with $J(n, k_0)$).

Note that the matrix $A' = \frac{1}{a}A$ belongs to $\mathrm{DS}(W')$ for some a. On the other hand $A' \notin \mathrm{Conv}(\mathrm{Aut}(W'))$, since the restriction of A' to the k_0th cell of W' does not belong to $\mathrm{Conv}(\mathrm{Aut}(W_0))$. Thus W' is not compact.■

Remark 5.6 *Let S be a nonempty subset of $[0,n]$. Denote by W'_S the centralizer algebra of the action of* Sym(n) *on the subsets of $[1,n]$ the cardinality of which belongs to S. Then W'_S is compact iff either $S \subset \{0, 1, n-1, n\}$ or $n = 4$ and $2 \in S$, $1, 3 \notin S$. Indeed,* *for $n \geq 5$ this was in fact proved in the theorem. For $n \leq 3$ it is trivial. If $n = 4$, then it suffices to check that the algebra $W_{\{1,2\}}$ is not compact. This can be done for instance by Fukuda's program [4].*

5.4. Let us turn into the compactness problem of the Platonic solids graphs. It is known (see [1]) that all of them are distance-regular (even distance-transitive) graphs. So their compacntess is equivalent to that of the corresponding cellular algebras (see subsection 3.1).

Theorem 5.7 *The graphs of the tetrahedron, the octahedron and the cube are compact, the graphs of the dodecahedron and the icosahedron are not compact.*

Proof. It is clear that the cellular algebra of the tetrahedron is isomorphic to $S(4)$, whence its compactness follows. The cellular algebras of the octahedron and the cube are isomorphic to $S(2) \wr S(3)$ and $S(4) \otimes S(2)$. Thus their compactness follows from theorem 4.2 and theorem 4.3 respectively.

Let us finally consider the dodecahedron and the icosahedron. In the both cases denote by W the corresponding cellular algebra and by E its antipodal equivalence the classes of which are pairs of vertices at maximal distance. In the case of the dodecahedron W/E is isomorphic to the cellular algebra of Petersen's graph which is not compact by corollary 5.4. So W is not compact by theorem 3.8. In the case of icosahedron W/E is isomorphic to $S(6)$. So the non-compactness of W also follows from theorem 3.8 after taking into account the fact that $|\operatorname{Aut}(W)| = 120$ whereas $|\operatorname{Aut}(W/E)| = 720$.

5.5. A transitive permutation group $G \leq \operatorname{Sym}(V)$ is called Frobenius group (see [9]) if it is not regular and $G_{u,v} = \{1\}$ for all distinct $u, v \in V$. By the Frobenius theorem G has a normal regular subgroup H called the Frobenius kernel of G. In [9] it was shown that H is Abelian if its index in G is even.

Theorem 5.8 *A Frobenius group $G \leq \operatorname{Sym}(V)$ is compact iff $|G_v| = 2$, $v \in V$.*

Proof. If $|G_v| = 2$, then the compactness of G follows from theorem 5.1 and the result cited above. Let $|G_v| \geq 3$. It follows from the definition that for all $k \in G$

$$\sum_{g \in Hk} P_g = J_V \quad \text{and} \quad P_g \circ P_{g'} = 0 \quad \Leftrightarrow \quad Hg = Hg' \tag{16}$$

where H is the Frobenius kernel of G. Since $|G_v| \geq 3$, there exist $k, k' \in G \setminus H$ such that $Hk \neq Hk'$ and $k, k' \in G_v$. Set

$$A = \sum_{g \in Hk \setminus \{k\}} P_g + P_{k'} - P_1.$$

By the choice of k and k' and (16) all the elements of A are non-negative integers and $A \circ P_1 = 0$. Let us check that for the pair (G, A) condition (15) is satisfied, whence the non-compactness of G will follow. Indeed, by (16) if $g \notin H \setminus \{1\}$, then $P_g \circ P_1 \neq 0$, else $P_g \circ P_k \neq 0$.∎

The theorem enables us to state that the minimal n for which a non-compact cellular algebra on n points exists, is equal to 7. Indeed, it implies that the semidirect product of cyclic groups of order 7 and 3, acting on 7 points is not compact. Since this group is 2-closed, its centralizer algebra is also not compact. On the other hand, due to [8] all cellular algebras on n points are Schurian for $n \leq 8$. So it suffices to check that all 2-closed permutation groups of degree at most 6 are compact. However the centralizer algebras of these groups can be constructed from simplexes, regular algebras and the algebras of undirected cycles by the compactness preserving operations described in sections 3 and 4.

6 Algorithms

6.1. Throughout the section we assume that $V = [1, n]$ and deal with cellular algebras W on V the basis matrices of which are numbered by positive integers (colors) $1, \ldots, s$ where $s = |\mathcal{R}(W)|$. The color of $v \in V$ with respect to W is defined to be the color of the matrix $I_U \in \mathcal{R}(W)$ where U is the cell of W containing v. Under isomorphism of such algebras we mean an ordinary cellular algebra isomorphism preserving the colors of the basis matrices.

Given a cellular algebra W on V and $A \in \mathrm{Mat}_V$, we put in order the set of the basis matrices of the algebra $W[A]$ according to the Weisfeiler-Lehman algorithm for constructing cellular closure, so that the following property holds (see [14, Ch.M]):

(W-L) if $g \in \mathrm{Sym}(V)$ is an isomorphism from W to W' and $A^g = A'$, then g is also an isomorphism from $W[A]$ to $W'[A']$.

The standard basis of $W[A]$ (with the order) can be constructed by this algorithm in polynomial time from W and A. In this way we put in order the basis matrices of the algebra W_v, $v \in V$, and inductively of the algebras $W_{v_1,\ldots,v_k} = (W_{v_1,\ldots,v_{k-1}})_{v_k}$.

6.2. Let us describe the k-orbits of the automorphism group of a compact cellular algebra on V, i.e. the orbits of the induced action of this group on V^k, $k \geq 1$.

Let W be a cellular algebra on V and $(c_1, \ldots, c_k)$ be a k-tuple of positive integers. We say that $(c_1, \ldots, c_k)$ is W-*admissible* if there exists a k-tuple $(v_1, \ldots, v_k) \in V^k$ such that for each $i \in [1, k]$ the color of v_i with respect to $W_{v_1,\ldots,v_{i-1}}$ equals c_i. The set of all these $(v_1, \ldots, v_k)$ is denoted by $S(c_1, \ldots, c_k)$. It is clear that $S(c_1, \ldots, c_k)$ is a union of the k-orbits of $\mathrm{Aut}(W)$.

Proposition 6.1 *Let $W \leq \mathrm{Mat}_V$ be a compact cellular algebra. A subset S of V^k is a k-orbit of $\mathrm{Aut}(W)$ iff $S = S(c_1, \ldots, c_k)$ for some W-admissible tuple $(c_1, \ldots, c_k)$.*

Proof. Let $(c_1, \ldots, c_k)$ be a W-admissible tuple and $(v_1, \ldots, v_k), (v'_1, \ldots, v'_k) \in S(c_1, \ldots, c_k)$. Let us show by induction on i that there exists $g_i \in \mathrm{Aut}(W)$ such that g_i is an isomorphism from $W_{v_1,\ldots,v_{i-1}}$ to $W_{v'_1,\ldots,v'_{i-1}}$. The induction base is provided by setting $g_1 = 1$. Let the isomorphism g_i be already constructed and $u = v_i^{g_i}$. Then the colors of u and v'_i with respect to $W_{v'_1,\ldots,v'_{i-1}}$ coincide by the definition of $S(c_1, \ldots, c_k)$ and due to the fact that g_i is an isomorphism of colored algebras. By theorem 3.7 and theorem 3.4 there exists $h \in \mathrm{Aut}(W_{v'_1,\ldots,v'_{i-1}})$ such that $u^h = v'_i$. According to (W-L) $g_{i+1} = g_i h$ is the required isomorphism from $W_{v_1,\ldots,v_i}$ to $W_{v'_1,\ldots,v'_i}$.

Conversely, let S be a k-orbit of $\mathrm{Aut}(W)$ and $(v_1, \ldots, v_k) \in S$. For each $i \in [1, k]$ define c_i to be the color of v_i with respect to $W_{v_1,\ldots,v_{i-1}}$. Then the tuple $(c_1, \ldots, c_k)$ is W-admissible and $(v_1, \ldots, v_k) \in S \cap S(c_1, \ldots, c_k)$. By the first part of the proof $S = S(c_1, \ldots, c_k)$.■

6.3. Proposition 6.1 enables to find a canonical labeling for the class $\mathcal{W}_V$ of compact cellular algebras on V, i.e. a map $cl : \mathcal{W}_V \to \mathrm{Sym}(V)$ such that the following condition is satisfied:

$$W_1 \cong W_2 \quad \Leftrightarrow \quad W_1^{\sigma_1} = W_2^{\sigma_2}, \qquad W_1, W_2 \in \mathcal{W}_V$$

where $\sigma_i = cl(W_i)$, $i = 1, 2$. This can be done by the following procedure:

$$\begin{array}{lll} \underline{\text{for}} & i & \in [1,n] \quad \underline{\text{do}} \\ & U & := \{v_1, \ldots, v_{i-1}\}; \\ & c & := \min_{v \in V \setminus U} c(v, W_{v_1,\ldots,v_{i-1}}); \\ & i^{\sigma^{-1}} & := \min\{v \in V : c(v, W_{v_1,\ldots,v_{i-1}}) = c\}; \\ \underline{\text{od}} & & \end{array}$$

where $v_j = j^{\sigma^{-1}}$ and $c(v, W_{v_1,\ldots,v_{i-1}})$ is the color of v with respect to $W_{v_1,\ldots,v_{i-1}}$. It is clear that the permutation $\sigma = cl(W)$ is computed in polynomial time.

The above algorithm shows the way to find the group $\mathrm{Aut}(W)$ for $W \in \mathcal{W}_V$ in polynomial time. If $W = \mathrm{Mat}_V$, then $\mathrm{Aut}(W) = \{1\}$. Otherwise, fix a point $v_0 \in V$ for which $W_{v_0} \neq W$ and construct a generator set of the group $\mathrm{Aut}(W)$ recursively from a generator set of $\mathrm{Aut}(W_{v_0})$. The recursion is provided by the equality

$$\mathrm{Aut}(W) = \bigcup_{v \in \Delta} \mathrm{Aut}(W_{v_0}) g_v$$

where $\Delta = \{v \in V : \ W_v \cong W_{v_0}\}$ and $g_v = cl(W_{v_0}) cl(W_v)^{-1}$. Note that we really find a strong generator set of $\mathrm{Aut}(W)$. Thus the following statement is proved.

Theorem 6.2 *A canonical labeling in the class of compact algebras (weakly compact graphs) and the automorphism group of a compact algebra (a weakly compact graph) can be found in polynomial time.*

It is well-known that the Graph Isomorphism Problem is equivalent to the Setwise Stabilizer Problem: given a permutation group G on V and a set $U \subset V$, find the subgroup of all permutations of G leaving U fixed as a set. Theorem 6.2 shows that in the class of compact groups this problem can be solved in polynomial time. Indeed, it suffices to find $\mathrm{Aut}(W[I_U])$ where $W = \mathcal{Z}(G)$. The compactness of the algebra $W[I_U]$ is provided by theorem 3.7.

7 Open problems

All primitive compact permutation groups which we know coincide with the natural representations of symmetric groups or dihedral and cyclic groups of prime degree.

Problem 7.1 *Are there any other compact primitive groups?*

It is a well-known fact that the structure constants of a cellular algebra do not determine it up to a cellular algebra isomorphism (see [2]). However, we have no such examples for compact algebras.

Problem 7.2 *Is it true that two weakly isomorphic compact cellular algebras are isomorphic?*

Here under a weak isomorphism of cellular algebras we mean an ordinary algebra isomorphism preserving the Hadamard multiplication and the Hermitian conjugation.

In [5] a polynomial-time algorithm for recognizing compact regular graphs with prime number of vertices was described. Certainly, it also works for homogeneous algebras of prime degree.

Problem 7.3 *Is there an efficient procedure to recognize general compact algebras?*

One can easily prove by using the technique of section 6 that this problem is polynomial-time equivalent to the problem of recognizing compact groups.

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