On a New High Dimensional Weisfeiler-Lehman Algorithm

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Abstract

In the paper we define and study a generalization of the Weisfeiler-Lehman algorithm which constructs the cellular closure of a set of matrices. The new technique is compared with all the other techniques of that kind. The underlying construction gives a new sufficient condition for a cellular algebra to be Schurian.

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1 Introduction

The starting point of the paper is the Graph Isomorphism Problem (ISO), a famous unsolved problem in computation complexity theory. The problem is to test whether two finite graphs are isomorphic by means of an efficient algorithm. Despite many efforts, at present the best isomorphism test for n-vertex graphs makes at least $n^{O(\sqrt{n})}$ steps in the worst case [4].

It is well-known [10] that the ISO is polynomially equivalent to the following problem: given a graph Γ construct all the orbits of its automorphism group $\mathrm{Aut}(\Gamma)$ consisting by definition of all permutations of the vertex set V of Γ, preserving the adjacency of edges. It is easy to see that the orbits of $\mathrm{Aut}(\Gamma)$ can efficiently be computed by a linear base of the centralizer algebra $\mathcal{Z}(\mathrm{Aut}(\Gamma), V)$ of the permutation group $(\mathrm{Aut}(\Gamma), V)$, consisting of all the matrices over $\mathbf{C}$ commuting with each permutation matrix of $\mathrm{Aut}(\Gamma)$. Thus the ISO is polynomially equivalent to the problem of finding the algebra $\mathcal{Z}(\mathrm{Aut}(\Gamma), V)$.

In [13] an approach to the ISO based on the notion of *cellular algebra* was developed. Denote by Mat_V the full matrix algebra over $\mathbf{C}$ on V, i.e. the set of all complex matrices whose rows and columns are indexed by the elements of V. A subalgebra of Mat_V is called cellular if it is closed under the Hadamard (componentwise) multiplication, the Hermitian conjugation and contains the matrix whose all the entries are equal to 1. One can see that the centralizer algebra $\mathcal{Z}(G, V)$ of a permutation group (G, V) is cellular. On the other hand, given a matrix $A \in \mathrm{Mat}_V$ the smallest cellular algebra containing A denoted by $[A]$ can be constructed in polynomial time [13]. It gives an efficient algorithm to construct the cellular algebra $W(\Gamma) = [A_\Gamma]$ of a graph Γ with adjacency matrix A_Γ.

It is proved in [13] that $\mathrm{Aut}(\Gamma)$ coincides with $\mathrm{Aut}(W(\Gamma))$ which is by definition the group of all permutation matrices of Mat_V commuting with each matrix of $W(\Gamma)$. This implies $W(\Gamma) \le \mathcal{Z}(\mathrm{Aut}(\Gamma), V)$ (inclusion). Note that the ISO could be solved if the last inequality was always equality. Unfortunately, there exist cellular algebras W not coinciding with $\mathcal{Z}(\mathrm{Aut}(W), V)$ denoted by $\mathrm{Sch}(W)$ below (see [1], [14]). These algebras were called in [9] non-Schurian in contrast to Schurian ones for which $W = \mathrm{Sch}(W)$.

The above paragraph explains why the property of a cellular algebra "to be Schurian" is extremely important in the context of the ISO. Indeed, it shows that the ISO is polynomially equivalent to the problem of constructing the Schurian closure $\mathrm{Sch}(W)$ of a cellular algebra W. Here we face a common situation in mathematics: we want to construct some object but have in hand only an approximation to it. Certainly, it would be more convenient to deal with a sequence of some natural approximations giving eventually the object we are interested in. This motivates the following definition of Schurian Polynomial Approximation Scheme.

Let us have a rule according to which given a cellular algebra $W \le \mathrm{Mat}_V$ and a positive integer m a cellular algebra $W^{(m)} \le \mathrm{Mat}_V$ can be constructed. We say that the operators $W \mapsto W^{(m)}$ $(m = 1, 2, \ldots)$ define a *Schurian Polynomial Approximation Scheme* if the following statements hold:

(1) $\mathrm{Aut}(W^{(m)}) = \mathrm{Aut}(W)$ for all $m \ge 1$;

(2) $W = W^{(1)} \le \ldots \le W^{(n)} = \ldots = \mathrm{Sch}(W)$;

(3) $(W^{(m)})^{(l)} = W^{(m)}$ for all $l \in [1, m]$;

(4) $W^{(m)}$ can be constructed in time $n^{O(m)}$

where $n = |V|$. In this paper we describe a special Schurian Polynomial Approximation Scheme and study its main properties.

The key notion of our aproach is that of m-closure. To define it given W and m denote by $W^{(m)}$ a cellular algebra which is the restriction to V (included in V^m diagonalwise) of the smallest cellular subalgebra of Mat_{V^m} containing the algebras $\mathcal{Z}(\mathrm{Sym}(V), V^m)$ and $W \otimes \cdots \otimes W$ (m times). We call $W^{(m)}$ the *m-closure* of W.

Theorem 1.1 *The m-closure operators $W \mapsto W^{(m)}$ $(m = 1, 2, \ldots)$ constructed above define a Schurian Polynomial Approximation Scheme.*

In [13], [5] and [6] some algorithms which can be interpreted in the above terms as ones defining Schurian Polynomial Approximation Schemes were described. In this interpretation we denote by A_m (resp. B_m) the algorithm from [5] (resp. from [6]). A natural question arises: what does a relationship exist between the cellular algebras $A_m(W)$, $B_m(W)$ and $W^{(m)}$? The next proposition gives a partial answer to the question.

Theorem 1.2 *The Schurian Polynomial Approximation Scheme defined by the m-closure operators "is not worse" than ones defined by A_m and B_m. More exactly,*

$$W^{(m)} \geq A_m(W), \quad W^{(m)} \geq B_m(W)$$

for all cellular algebras W and $m \geq 1$.

One of the most important problem concerning Schurian Polynomial Approximation Schemes is a good estimation of the smallest $m \geq 1$ for which $W^{(m)} = \mathrm{Sch}(W)$. Note, that if such m was bounded by a constant then by (4) the ISO could be solved in polynomial time. We don't know whether this is true for our or someone's else Schurian Polynomial Approximation Scheme. However, we can give an upper bound for m in terms of the split number of a cellular algebra defined below.

The split number $s(W)$ of a cellular algebra W is by definition the smallest s for which there exist $v_1, \ldots, v_s \in V$ such that $W_{v_1,\ldots,v_s} = \mathrm{Mat}_V$ where $W_{v_1,\ldots,v_s} = W[I_{v_1}, \ldots, I_{v_s}]$ is the smallest cellular subalgebra of Mat_V containing W and all the matrices I_{v_i}, I_{v_i} being a {0,1}-matrix with exactly one nonzero element standing in row and column v_i. Clearly, $s(W) \leq n - 1$ for all W. Some non-trivial upper bounds for this number can be found in [3] and [11]. We also mention paper [7] where a similar invariant (called freedom degree) was defined for a permutation group.

Theorem 1.3 *Let W be a cellular algebra with $s(W) \leq m - 1$. Then* $\mathrm{Sch}(W) = W^{(m)}$.

The idea of the proof is to consider a cellular algebra $\widehat{W}^{(m)} \subset \mathrm{Mat}_{V^m}$ generated by the algebras $\mathcal{Z}(\mathrm{Sym}(V), V^m)$ and $W \otimes \cdots \otimes W$ (m times). We show that this algebra contains (in a sense) all cellular algebras of the form $W_{v_1,\ldots,v_m}$ with $v_i \in V$. By using this fact we find a faithful regular orbit of the permutation group $(\mathrm{Aut}(W), V^m) = (\mathrm{Aut}(\widehat{W}^{(m)}), V^m)$. A final step is to transfer the corresponding action of $\mathrm{Aut}(W)$ to V.

As an easy corollary ($s(W) \leq 1$) we get the following statement.

Corollary 1.4 *If a cellular algebra $W \leq \mathrm{Mat}_V$ has no proper cellular superalgebras, then $W^{(2)} = \mathrm{Sch}(W)$.*

The paper consists of six sections. The second one contains main definitions and some preliminary results concerning cellular algebras. In section 3 we define the notion of m-closure and in detail study the properties of m-closed cellular algebras. As a result we get the proof of Theorem 1.1. Sections 4 and 5 are devoted to Theorems 1.2 and 1.3 respectively. In section 6 we discuss some open problems.

Notations. As usual by $\mathbf{C}$ we denote the field of all complex numbers.

Throughout the paper V denotes a finite set with $n = |V|$ elements. The group of all permutations of V is denoted by $\mathrm{Sym}(V)$. By relations on V me mean subsets of $V \times V$. If R is a relation on V, then

$$\mathrm{supp}(R) = \bigcap_{U, R \subset U \times U} U.$$

An equivalence E is by definition a symmetric, transitive, but non-necessary reflexive relation on V. In this case $V/E = \mathrm{supp}(E)/E$ denotes the set of all equivalence classes modulo E.

The algebra of all complex $n \times n$ matrices whose rows and columns are indexed by the elements of V is denoted by Mat_V. If $A \in \mathrm{Mat}_V$, then A^T denotes the transpose and A^* the Hermitian conjugate matrix.

For a set M

$$I_M = \{(i,i) |\ i \in M\}, \quad J_M = \{(i,j) |\ i,j \in M\}.$$

For integers l, m with $l \le m$ by $[l, m]$ we denote the set $\{l, l+1, \ldots, m\}$.

2 Cellular algebras

Denote by L_V a linear space over $\mathbf{C}$ with the set V as a base. For any subset $U \subset V$ the linear space L_U can naturally be viewed as a subspace of L_V (spanned by U). Below we identify L_U with this subspace of L_V.

By a *cellular algebra* W on V we mean a subalgebra of Mat_V containing the identity matrix I_V, the matrix J_V whose all the entries are equal to 1, and closed under the Hermitian conjugation and the Hadamard (componentwise) multiplication (denoted by $\circ$ below). The algebra Mat_V naturally acts on the space L_V. The restriction of this action to W defines a faithful linear representation called the *standard representation* of W.

Below we give a combinatorial characterization of cellular algebras. It is convenient to view {0,1}-matrices belonging to Mat_V as the adjacency matrices of relations on V. Throughout the paper we identify these matrices with the corresponding relations.

Proposition 2.1 ([13]) *A linear subspace $W \subset \mathrm{Mat}_V$ is a cellular algebra if and only if there exists a linear basis $\mathcal{R} = \mathcal{R}(W)$ of W consisting of {0,1}-matrices such that*

(1) $\sum_{R \in \mathcal{R}} R = J_V$;

(2) $R \in \mathcal{R} \leftrightarrow R^T \in \mathcal{R}$;

(3) there exists a disjoint partition $V = \bigcup_{i=1}^{s} V_i$ *of V into nonempty sets V_i such that*

  (a) $I_{V_i} \in \mathcal{R}$ *for all i;*

(b) $\forall R \in \mathcal{R}\ \exists i, j \in [1, s]:\ R \subset V_i \times V_j$*;*

(c) if $R \in \mathcal{R}$ *and* $R \subset V_i \times V_j$*, then the number of 1's in the* u*th row (resp.* v*th column) of the matrix* R *does not depend on the choice of* $u \in V_i$ *(resp.* $v \in V_j$*), this number is denoted by* $d_{\rm out}(R)$ *(resp.* $d_{\rm in}(R)$*);*

(4) given $R, S, T \in \mathcal{R}$ *the number*

$$p(u, v; S, T) = \Big|\{w \in V \mid (u, w) \in S, (w, v) \in T\}\Big|, \quad u, v \in V$$

does not depend on the choice of $(u, v) \in R$*.*■

Remark 2.2 *It is easily seen that the basis* $\mathcal{R}$ *and the partition* $V = \bigcup_{i=1}^{s} V_i$ *are uniquely determined by* W*.*

The linear basis $\mathcal{R}(W)$ of a cellular algebra W defined in Proposition 2.1 is called the *standard basis* of W. Any subset $V_i \subset V$ defined in Proposition 2.1 (resp. a union possibly empty of V_i) is called a *cell* (resp. a *cellular set*) of W. The set of all of them is denoted by $\mathrm{Cel}(W)$ (resp. $\mathrm{Cel}^*(W)$).

Below we will use the following generalization of statement (4) of Proposition 2.1. Let $u, v \in V$ and $\delta = (R_1, \ldots, R_l) \in \mathcal{R}^l$. We say that $(v_0, \ldots, v_l) \in V^{l+1}$ is a (u, v)*-path of the type* δ if $v_0 = u$, $v_l = v$ and $(v_{i-1}, v_i) \in R_i$ for all $i \in [1, l]$. The number of all such paths will be denoted by $p(u, v; \delta)$.

Lemma 2.3 (Path Proposition [13], Th. C10) *Let* W *be a cellular algebra. Then given* $R \in \mathcal{R}(W)$ *the integer* $p(u, v; \delta)$ *does not depend on the choice of* $(u, v) \in R$*.*■

Let $U \in \mathrm{Cel}^*(W)$ be a cellular set. The subalgebra $I_U W I_U \subset W$ invariantly acts on the subspace $L_U = I_U L_V \subset L_V$. So it can be viewed as a subalgebra of Mat_U. Clearly, it is closed under the Hermitian conjugation and the Hadamard multiplication and contains I_U and J_U. Thus it is a cellular algebra on U called the *restriction* of W to U and denoted by W_U.

The set of all cellular algebras on V is ordered by inclusion. The algebra Mat_V is obviously the greatest element of the set. We write $W \le W'$ if W is a subalgebra of W'. If $A_1, \ldots, A_m \in \mathrm{Mat}_V$, then the intersection of all cellular algebras on V containing W and all the matrices A_i is also a cellular algebra on V. It is denoted by $W[A_1, \ldots, A_m]$. We use notation $[A_1, \ldots, A_m]$ if W is a simplex i.e. $\mathcal{R}(W) = \{I_V, J_V - I_V\}$, and $W_{v_1, \ldots, v_m}$ if $A_i = I_{v_i} = I_{\{v_i\}}$ for all i.

Let $\varphi : V \to V'$ be a bijection. It induces in a natural way a linear isomorphism $L_V \to L_{V'}$ and an algebra isomorphism $\mathrm{Mat}_V \to \mathrm{Mat}_{V'}$ such that

$$(Ax)^{\varphi} = A^{\varphi} \cdot x^{\varphi}, \qquad A \in \mathrm{Mat}_V,\ x \in L_V.$$

Two cellular algebras W on V and W' on V' are called *isomorphic* if $W^{\varphi} = W'$ (as sets) for some bijection $\varphi : V \to V'$ called an *isomorphism* from W to W'. Clearly, φ induces a bijection between the sets $\mathcal{R}(W)$ and $\mathcal{R}(W')$. The group of all isomorphisms from W to itself contains a normal subgroup

$$\mathrm{Aut}(W) = \{\varphi \in \mathrm{Sym}(V) \mid A^{\varphi} = A,\ A \in W\}$$

called the *automorphism group* of W.

Following [13] let us define for cellular algebras the notion of tensor product. Let $W_1 \leq \mathrm{Mat}_{V_1}$ and $W_2 \leq \mathrm{Mat}_{V_2}$ be cellular algebras on V_1 and V_2. It is easy to see that the subalgebra $W_1 \otimes W_2$ of $\mathrm{Mat}_{V_1} \otimes \mathrm{Mat}_{V_2} = \mathrm{Mat}_{V_1 \times V_2}$ is closed under the Hadamard multiplication in $\mathrm{Mat}_{V_1 \times V_2}$. It also contains $I_{V_1 \times V_2} = I_{V_1} \otimes I_{V_2}$ and $J_{V_1 \times V_2} = J_{V_1} \otimes J_{V_2}$. So $W_1 \otimes W_2$ is a cellular algebra on $V_1 \times V_2$ called the tensor product of W_1 and W_2. Clearly, $\mathcal{R}(W_1 \otimes W_2) = \mathcal{R}(W_1) \otimes \mathcal{R}(W_2)$ and $\mathrm{Aut}(W_1 \otimes W_2) = \mathrm{Aut}(W_1) \times \mathrm{Aut}(W_2)$.

A large class of cellular algebras comes from permutation groups as follows (see [13]). Let (G, V) be a permutation group. Then its centralizer algebra $\mathcal{Z}(G, V) \subset \mathrm{Mat}_V$ is a cellular algebra on V the standard basis of which consists of all the orbits of the natural action of G on $V \times V$. For a cellular algebra W on V set

$$\mathrm{Sch}(W) = \mathcal{Z}(\mathrm{Aut}(W), V).$$

Clearly, $W \leq \mathrm{Sch}(W)$ and $\mathrm{Aut}(W) = \mathrm{Aut}(\mathrm{Sch}(W))$. The algebra W is called *Schurian* if $W = \mathrm{Sch}(W)$. Certainly, $\mathrm{Sch}(W)$ is a Schurian algebra for all W. It follows from [14, 1] that there exist cellular algebras which are not Schurian. It is well-known that the ISO is polynomially equivalent to the problem of constructing $\mathrm{Sch}(W)$.

The isomorphism of cellular algebras W and W' defined above induces an equivalence between the standard representations of them. The converse statement is not true. This motivates the following definition (see [13]). Cellular algebras W on V and W' on V' are called *weakly isomorphic* if there exists an algebra isomorphism $f : W \to W'$ such that $f(\mathcal{R}(W)) = \mathcal{R}(W')$. Any such f is called a *weak isomorphism* from W to W'. The following statement describes the basic properties of weak isomorphisms which are easily deduced from the definition.

Proposition 2.4 *Let $f : W \to W'$ be a weak isomorphism. Then*

(1) $\forall A, B \in W: \quad f(A \circ B) = f(A) \circ f(B), \quad f(A^*) = f(A)^*.$

(2) $\forall U \in \mathrm{Cel}(W)\ \exists U' \in \mathrm{Cel}(W'): \ |U| = |U'|, \ f(I_U) = I_{U'}$. *In particular,* $|V| = |V'|$, $|\mathrm{Cel}(W)| = |\mathrm{Cel}(W')|$.

(3) $E \in W$ *is an equivalence on* V *iff* $E' = f(E) \in W'$ *is an equivalence on* V'. *Moreover,* $|V/E| = |V/E'|$ *and* $\{|U|;\ U \in V/E\} = \{|U'|;\ U' \in V'/E'\}$.■

Let W be a cellular algebra on V. A nonempty equivalence $E \in W$ on V is called *indecomposable* (in W) if E is not a matrix sum of two nonempty equivalences on V belonging to W. Otherwise, the equivalence is called *decomposable*. Since $E \supset I_{V_E}$ for any equivalence $E \in W$ where $V_E = \mathrm{supp}(E) \in \mathrm{Cel}^*(W)$, each equivalence belonging to W can uniquely be represented as a sum of indecomposable ones called *indecomposable components* of E.

Let $E \in W$ be an equivalence on V. For each $U \in V/E$ the set $W_{E,U} = I_U W I_U$ can be viewed as a cellular algebra on U with the standard basis

$$\mathcal{R}(W_{E,U}) = \{I_U R I_U \mid R \in \mathcal{R},\ R \subset E,\ I_U R I_U \neq 0\}. \tag{1}$$

Clearly, each basis relation of $W_{E,U}$ can uniquely be represented in the form $I_U R I_U$ with $R \in \mathcal{R}$.

Lemma 2.5 *If E is an indecomposable equivalence of W, then*

(1) given $U,U' \in V/E$ there exists a uniquely defined weak isomorphism

$$f_{U,U'} : W_{E,U} \to W_{E,U'}$$

such that $f_{U,U'}(I_U A I_U) = I_{U'} A I_{U'}$ for all $A \in W$.

(2) $\forall U,U' \in V/E\ \forall V_i \in \mathrm{Cel}(W),\ V_i \subset \mathrm{supp}(E) : \quad |U \cap V_i| = |U' \cap V_i| > 0.$ ■

Proof. We define $f = f_{U,U'}$ as follows. Let $S \in \mathcal{R}(W_{E,U})$. By (1) there exists a uniquely determined $R \in \mathcal{R}(W), R \subset E$ such that $S = I_U R I_U$. Set $f(S) = I_{U'} R I_{U'}$. We will show that f is a bijection from $\mathcal{R}(W_{E,U})$ to $\mathcal{R}(W_{E,U'})$. By (1) it suffices to check that

$$\forall R \in \mathcal{R}(W) : \quad I_U R I_U = 0 \ \leftrightarrow\ I_{U'} R I_{U'} = 0 \tag{2}$$

Indeed, if $I_U R I_U = 0$ and $I_{U'} R I_{U'} \neq 0$, then $V_R \subset V_E$, $V_R \neq \emptyset$ and $V_R \cap U = \emptyset$ where $V_R = \mathrm{supp}(R)$ and $V_E = \mathrm{supp}(E)$. So E is a sum of two nonempty equivalences of W: $I_{V_R} E I_{V_R}$ and $I_{V_E \setminus V_R} E I_{V_E \setminus V_R}$, which contradicts the indecomposability of E.

Extend f to a linear map from $W_{E,U}$ to $W_{E,U'}$. This map is an algebra isomorphism, since

$$f(I_U R_1 I_U \cdot I_U R_2 I_U) = f(I_U R_1 R_2 I_U) = I_{U'} R_1 R_2 I_{U'} = I_{U'} R_1 I_{U'} \cdot I_{U'} R_2 I'_U$$

for all $R_1, R_2 \in \mathcal{R}(W),\ R_1, R_2 \subset E$.

Let us prove statement (2). Since $V_i \subset \mathrm{supp}(E)$, there exists $U \in V/E$ for which $|U \cap V_i| > 0$. By using (2) for $R = I_{V_i}$ we conclude that $|U' \cap V_i| > 0$ for all $U' \in V/E$. Now statement (2) follows, since $U \cap V_i$ and $U' \cap V_i$ are blocks of the equivalence $I_{V_i} E I_{V_i}$ of the cellular algebra W_{V_i} with exactly one cell. ■

3 Extended algebras and m-closures

Let W be a cellular algebra on V. For each positive integer m we set

$$\widehat{W} = \widehat{W}^{(m)} = [\underbrace{W \otimes \cdots \otimes W}_{m},\ \mathcal{Z}(\mathrm{Sym}(V), V^m)]$$

with $\mathrm{Sym}(V)$ acting on V^m in a natural way:

$$(v_1, \ldots, v_m)^g = (v_1^g, \ldots, v_m^g), \quad g \in \mathrm{Sym}(V).$$

We call the cellular algebra $\widehat{W}^{(m)} \leq \mathrm{Mat}_{V^m}$ the *m-dimension extended algebra* of W. Clearly, $W^{(1)} = W$ and

$$\mathrm{Aut}(\widehat{W}^{(m)}) = \{\underbrace{(g, \ldots, g)}_{m} \mid g \in \mathrm{Aut}(W)\} \tag{3}$$

for all m.

Now we are going to describe some relations belonging to $\widehat{W}$. To do this we define for an arbitrary $S \subset [1,m]^2$ a binary relation P_S on V^m as follows:

$$(\bar{u},\bar{v}) \in P_S \;\leftrightarrow\; \forall (i,j) \in S: \; u_i = v_j \tag{4}$$

where $\bar{u} = (u_1,\ldots,u_m), \bar{v} = (v_1,\ldots,v_m) \in V^m$. Clearly,

$$P_S \in \mathcal{Z}(\mathrm{Sym}(V), V^m) \quad \text{for all} \quad S \subset [1,m]^2.$$

Examples. Let $M \subset [1,m]$.

1. Set

$$D_M = P_S \quad \text{where} \quad S = J_M \cup I_{[1,m]\setminus M}. \tag{5}$$

Clearly, $D_M \subset I_{V^m}$ for all M, $D_\emptyset = I_{V^m}$ and $D_{[1,m]} = I_\Delta$ where

$$\Delta = \{(v,\ldots,v\} \in V^m |\; v \in V\}. \tag{6}$$

2. Set

$$E_M = P_S \quad \text{where} \quad S = I_M. \tag{7}$$

Clearly, E_M is an equivalence on V^m for all M and $E_\emptyset = J_{V^m}$, $E_{[1,m]} = I_{V^m}$.

Below we will mainly use the relations D_M and E_M as well as matrices

$$\widehat{A} = \underbrace{I_V \otimes \cdots \otimes I_V}_{m-1} \otimes A, \quad A \in W \tag{8}$$

also belonging to $\widehat{W}^{(m)}$.

Each class U of the equivalence $E_{[1,m-1]}$ is of the form

$$U = U_{v_1,\ldots,v_{m-1}} = \{(v_1,\ldots,v_{m-1},v) \mid v \in V\}$$

for some $v_i \in V$. Let us define a map ζ_U as follows:

$$\zeta_U : V \to U, \quad v \mapsto (v_1,\ldots,v_{m-1},v).$$

The following lemma describes the simplest properties of the map.

Lemma 3.1 *In the above notation the following statements hold:*

(1) ζ_U *is a bijection;*

(2) $R^{\zeta_U} = I_U \widehat{R} I_U = I_U \widehat{R} = \widehat{R} I_U$*;*

(3) $(W_{v_1,\ldots,v_{m-1}})^{\zeta_U} \le \widehat{W}_{E,U}$ *where* $\widehat{W} = \widehat{W}^{(m)}$ *and* $E = E_{[1,m-1]}$.

Proof. Statements (1) and (2) are trivial. By (2) $W^{\zeta_U} \le \widehat{W}_{E,U}$. On the other hand,

$$(I_{v_i})^{\zeta_U} = I_{(v_1,\ldots,v_{m-1},v_i)} = I_U D_{\{i,m\}} I_U \in \widehat{W}_{E,U} \quad \text{for all} \quad i \in [1,m-1].$$

Thus

$$\widehat{W}_{E,U} \ge W^{\zeta_U}[I_{v_1}^{\zeta_U},\ldots,I_{v_{m-1}}^{\zeta_U}] = (W_{v_1,\ldots,v_{m-1}})^{\zeta_U}. \blacksquare$$

For $l \in [1,m]$ define another map

$$\xi_l^m : V^l \to V^m, \quad (v_1,\ldots,v_l) \mapsto (v_1,\ldots,v_l,\ldots,v_l). \tag{9}$$

It is easy to see that ξ_l^m is an injection and $\xi_l^m(V^l) = \mathrm{supp}(D_{[l,m]})$ is a cellular set of $\widehat{W}^{(m)}$.

The important feature of the cellular algebra $\widehat{W}^{(m)}$ is the possibility to extend the algebra W without changing its automorphism group. To show it set

$$W^{(m)} = ((\widehat{W}^{(m)})_\Delta)^{\xi^{-1}}.$$

where $\xi = \xi_1^m : V \to V^m$ is the injection (9) and $\Delta = \xi(V)$ is the cellular set (6). Clearly, $W^{(m)} \geq W$ and $\mathrm{Aut}(W^{(m)}) = \mathrm{Aut}(W)$ (see (3)). We say that W is *m-closed* if $W = W^{(m)}$. Each algebra is certainly 1-closed. However it is not the case for $m \geq 2$. In fact we will show later that a non-Schurian cellular algebra cannot be m-closed for all $m \geq 2$.

Below we list some properties of the operators $\widehat{W} \mapsto \widehat{W}^{(m)}$, $W \mapsto W^{(m)}$.

Lemma 3.2 *For all cellular algebras W, W_1, W_2 on V and a positive integer m*

(1) $W_1 \leq W_2$ implies $\widehat{W}_1^{(m)} \leq \widehat{W}_2^{(m)}$ and $W_1^{(m)} \leq W_2^{(m)}$;

(2) $(\widehat{W_1 \cap W_2})^{(m)} \leq \widehat{W}_1^{(m)} \cap \widehat{W}_2^{(m)}$, $(W_1 \cap W_2)^{(m)} \leq W_1^{(m)} \cap W_2^{(m)}$;

(3) the intersection of m-closed cellular algebras is m-closed;

(4) $(\widehat{W}^{(l)})^{\xi_l^m} \leq (\widehat{W}^{(m)})_U$ for all $l \in [1,m]$ where $U = \xi_l^m(V^l)$ (see (9));

(5) $W^{(m)}$ is l-closed for all $l \in [1,m]$.

Proof. Statement (1) is clear. (2) follows from (1). If $W_1^{(m)} = W_1$ and $W_2^{(m)} = W_2$, then $(W_1 \cap W_2)^{(m)} \leq W_1 \cap W_2$ by (2). Since the inverse inclusion is obvious, (3) follows. To prove (4) it suffices to show that

$$(\mathcal{Z}(\mathrm{Sym}(V),V^l))^{\xi_l^m} \subset \mathcal{Z}(\mathrm{Sym}(V),V^m), \qquad (\underbrace{W \otimes \cdots \otimes W}_{l})^{\xi_l^m} \subset \underbrace{W \otimes \cdots \otimes W}_{m}.$$

The first is clear. The second one follows from $(A_1 \otimes \cdots \otimes A_l)^{\xi_l^m} = A_1 \otimes \cdots \otimes A_l \otimes \cdots \otimes A_l$.

Let us prove (5). It follows from statement (4) and $\xi_l^m \circ \xi_1^l = \xi_1^m$ (see (9)) that $W'^{(l)} \leq W'^{(m)}$ for all W'. Applying it to $W' = W^{(m)}$ we see that it suffices to prove statement (5) for $l = m$. We will check that the m-dimension extended algebras of W and $W^{(m)}$ coincide. Clearly, the second contains the first. To prove the inverse inclusion set $R_j = P_{S_j}$ where $S_j = \{(i,j) | \, i \in [1,m]\}$, $j \in [1,m]$ (see (4)). A straithforward calculation in $\mathrm{Mat}_{V^m} = \mathrm{Mat}_V \otimes \cdots \otimes \mathrm{Mat}_V$ shows that for all $j \in [1,m]$

$$R_j^T A^\xi R_j = \underbrace{J_V \otimes \ldots \otimes J_V}_{j-1} \otimes A \otimes \underbrace{J_V \otimes \ldots \otimes J_V}_{m-j}, \qquad A \in \mathrm{Mat}_V$$

where ξ is the map (9). Since the Hadamard multiplication in $\mathrm{Mat}_V \otimes \cdots \otimes \mathrm{Mat}_V$ can be done factorwise,

$$A_1 \otimes \cdots \otimes A_m = (R_1^T A_1^\xi R_1) \circ \cdots \circ (R_m^T A_m^\xi R_m) \quad \text{for all} \quad A_1,\ldots,A_m \in \mathrm{Mat}_V\,.$$

Thus $W^{(m)} \otimes \cdots \otimes W^{(m)} \subset \widehat{W}^{(m)}$, which completes the proof.■

It follows from statement (5) of Lemma 3.2 that the cellular algebra $W^{(m)}$ is m-closed. We call it the *m-closure* of W.

The following proposition describes a relationship between the notions of m-closure and Schurian closure $\mathrm{Sch}(W)$ of a cellular algebra W. It shows that in a sense $W^{(m)}$ can be interpreted as an approximation to $\mathrm{Sch}(W)$.

Proposition 3.3 *For each cellular algebra W on V the following statements hold:*

(1) $\mathrm{Aut}(W^{(m)}) = \mathrm{Aut}(W)$ *for all* $m \geq 1$*;*

(2) $W = W^{(1)} \leq \ldots \leq W^{(n)} = \ldots = \mathrm{Sch}(W)$*;*

(3) $(W^{(m)})^{(l)} = W^{(m)}$ *for all* $l \in [1, m]$.

Proof. Statement (1) is clear. Let us prove (2). The inclusion $W^{(l)} \leq W^{(m)}$ for $l \leq m$ is contained in the proof of statement (5) of Lemma 3.2. The equality $W^{(m)} = \mathrm{Sch}(W)$ for $m \geq n$ follows from Theorem 1.3, since clearly $s(W) \leq n-1$ for all W. (Note that Theorem 1.3 is proved in section 5 independently of this assertion.) Finally, (3) coincides with statement (5) of Lemma 3.2. ■

Proposition 3.4 *Given a cellular algebra W on V and a positive integer m the standard bases of the cellular algebras $\widehat{W}^{(m)}$ and $W^{(m)}$ can be constructed in time $n^{O(m)}$.*

Proof. Since the standard bases of $W \otimes \cdots \otimes W$ (m times) and $\mathcal{Z}(\mathrm{Sym}(V), V^m)$ can be found in time $n^{O(m)}$, the standard basis of $\widehat{W}^{(m)}$ (and so of $W^{(m)}$) can be found within the same time due to the Weisfeiler-Lehman algorithm for constructing the cellular closure of a set of matrices [13] (for a time analysis see also [12]).■

Remark 3.5 *If instead of the Weisfeiler-Lehman algorithm for constructing the cellular closure of a set of matrices we use an algorithm of [2], the algebras $\widehat{W}^{(m)}$ and $W^{(m)}$ can be found in time $O(mn^{3m}\log n)$.*

Propositions 3.3 and 3.4 show that the operators $W \mapsto W^{(m)}$ ($m = 1, 2, \ldots$) define a Schurian Polynomial Approximation Scheme (see Section 1). It proves Theorem 1.1.■

We complete the section by a statement being of use later. For each $R \subset V^2$ set

$$U_R = \{(u, \ldots, u, v) \in V^m |\ (u, v) \in R\}.$$

Proposition 3.6 *Let W be a cellular algebra on V and $m \geq 2$. Then*

(1) $\forall R \subset V^2: \quad R \in \mathcal{R}(W^{(m)}) \ \leftrightarrow \ U_R \in \mathrm{Cel}(\widehat{W}^{(m)})$*;*

(2) $\forall U \in \mathrm{Cel}(\widehat{W}^{(m)})\, \forall\, i, j \in [1, m]\, \exists R \in \mathcal{R}(W^{(m)}): \ ((v_1, \ldots, v_m) \in U \to (v_i, v_j) \in R)$.

Proof. Below we write $v_1 \ldots v_m$ instead of $(v_1, \ldots, v_m)$. Let us prove (1). Assume that $R \in \mathcal{R}(W^{(m)})$. Choose $(u, v) \in R$ and denote by S_1, T, S_2 the basis relations of $\widehat{W}^{(m)}$ containing the pairs $(u^m, u^{m-1}v), (u^{m-1}v, u^{m-1}v)$ and $(u^{m-1}v, v^m)$ respectively. Clearly, $p(u^m, v^m; \delta) = 1$ where $\delta = (S_1, T, S_2)$. By the Path Proposition (Lemma 2.3) the equality holds for all $(u')^m, (v')^m$ with $(u', v') \in R$. So $T = I_{U_R}$, whence $U_R \in \mathrm{Cel}(\widehat{W}^{(m)})$.

Conversely, let $U_R \in \mathrm{Cel}(\widehat{W}^{(m)})$. Choose $u^{m-1}v \in U_R$ and denote by S_1', R', S_2' the basis relations of $\widehat{W}^{(m)}$ containing the pairs $(u^{m-1}v, u^m), (u^m, v^m)$ and $(v^m, u^{m-1}v)$ respectively. Clearly, $p(u^{m-1}v, u^{m-1}v; \delta') = 1$ where $\delta' = (S_1', R', S_2')$. By the Path Proposition the equality holds for all points of U_R. It follows that $R' = R^\xi$ where ξ is defined in (9). That is $R \in \mathcal{R}(W^{(m)})$.

To prove (2) we assume without loss of generality that $i = m-1$, $j = m$. Let $U \in \mathrm{Cel}(\widehat{W}^{(m)})$. Choose $\bar{v} = v_1 \cdots v_m \in U$ and denote by R the basis relation of $W^{(m)}$ containing the pair (v_{m-1}, v_m). By (1) we have $U_R \in \mathrm{Cel}(\widehat{W}^{(m)})$. Set

$$S = (U_R \times U) \cap E_{\{m-1,m\}}$$

where $E_{\{m-1,m\}}$ is defined in (7). Clearly, $S \in \mathcal{R}(\widehat{W}^{(m)})$, $d_{\mathrm{in}}(S) = 1$ and $(v_{m-1}^{m-1}v_m, \bar{v}) \in S$. So for any $\bar{v}' \in U$ there exists $\bar{u}' \in U_R$ such that $(\bar{u}', \bar{v}') \in S$. If $\bar{v}' = v_1' \cdots v_m'$, then $\bar{u}' = (v_{m-1}')^{m-1}v_m'$, whence $(v_{m-1}', v_m') \in R$. ■

4 High dimensional Weisfeiler-Lehman procedures

In this section we prove Propositions 4.1 and 4.2 from which Theorem 1.2 follows.

A map f from V^m on $[1,d]$ is called a *coloring of* V^m *in* d *colors*. Any set $f^{-1}(i) \subset V^m$ is called a *color class* of f. Given $u, v \in V$ define a relation $R_f(u,v) \subset V \times V$ as follows:

$$R_f(u,v) = \{(u',v') \in V^2 |\ f(u', \ldots, u', v') = f(u, \ldots, u, v)\}.$$

Denote by $\mathcal{R}_f$ the set of all distinct $R_f(u,v)$. In this notation the following procedure was described in [13] (see also [5]).

Procedure A_m $(m \geq 2)$

Input: a cellular algebra W on V.
Output: a cellular algebra $A_m(W) \geq W$.
Step 1. Construct a coloring f_0 of V^m such that

$$f_0(\bar{v}) = f_0(\bar{v}') \quad \leftrightarrow \quad \forall R \in \mathcal{R}(W)\ \forall i,j \in [1,m]:\ ((v_i, v_j) \in R \quad \leftrightarrow \quad (v_i', v_j') \in R).$$

Set $l = 0$.
Step 2. For each $\bar{v} \in V^m$ find a formal sum $S(\bar{v}) = \sum_{u \in V} f_l(\bar{v}/u)$ where

$$\bar{v}/u = (\bar{v}_{1,u}, \ldots, \bar{v}_{m,u}) \quad \text{with} \quad \bar{v}_{i,u} = (v_1, \ldots, v_{i-1}, u, v_{i+1}, \ldots, v_m)$$

and

$$f_l(\bar{v}/u) = (f_l(\bar{v}_{1,u}), \ldots, f_l(\bar{v}_{m,u})).$$

Step 3. Find a coloring f_{l+1} of V^m such that

$$f_{l+1}(\bar{v}) = f_{l+1}(\bar{v}') \quad \leftrightarrow \quad (f_l(\bar{v}) = f_l(\bar{v}'),\ S(\bar{v}) = S(\bar{v}')).$$

If f_l and f_{l+1} have the different number of colors, then $l := l+1$ and go to Step 2. Otherwise set $f = f_l$ and $A_m(W) = [\mathcal{R}_f]$.■

Proposition 4.1 *Let W be a cellular algebra on V and $m \geq 2$. Then $W^{(m)} \geq A_m(W)$.*

Proof. We will show by induction on l that each color class of f_l is a union of the cells of the algebra $\widehat{W}^{(m)}$. Then given $R \in \mathcal{R}(W^{(m)})$, by statement (1) of Proposition 3.6 $f(\bar{v}) = f(\bar{v}')$ for all $\bar{v}, \bar{v}' \in U_R$ and we are done.

By statement (2) of Proposition 3.6 and the fact that $W \leq W^{(m)}$ it is true for $l = 0$. Suppose it holds for all $k < l$. Let $\bar{v} \in V^m$. For each $u \in V$ set

$$P_u(\bar{v}) = (\bar{v}, \bar{v}_{1,u}, \ldots, \bar{v}_{m,u}, \bar{v}).$$

It is easy to see that the path $P_u(\bar{v})$ from $\bar{v}$ to itself is of the type $\delta = (R_0, \ldots, R_m)$ for some basis relations $R_i \subset P_{S_i}$, (see (4)) where

$$S_i = \{(j,j), (i, i+1) \in [1,m]^2 \mid j \neq i, j \neq i+1\}, \quad i \in [0,m].$$

Moreover, any $(\bar{v}, \bar{v})$-path of the type δ coincides with $P_u(\bar{v})$ for some $u \in V$.

Let $\bar{v}, \bar{v}' \in V^m$ belong to the same cell of $\widehat{W}^{(m)}$. Then by the induction hypothesis $f_{l-1}(\bar{v}) = f_{l-1}(\bar{v}')$. Besides by the Path Proposition (Lemma 2.3) $p(\bar{v}, \bar{v}; \delta) = p(\bar{v}', \bar{v}'; \delta)$. If $P_u(\bar{v})$ and $P_{u'}(\bar{v}')$ are of the type δ, then $\bar{v}_{i,u}$ and $\bar{v}'_{i,u'}$ belong to the same cell of $\widehat{W}^{(m)}$ for all i. So by the induction hypothesis $f_{l-1}(\bar{v}/u) = f_{l-1}(\bar{v}'/u')$. Thus $S_{l-1}(\bar{v}) = S_{l-1}(\bar{v}')$ and consequently $f_l(\bar{v}) = f_l(\bar{v}')$.∎

Another implementation of the m-dimension procedure was described in [6]. We are going to prove that this procedure constructs a cellular subalgebra of the m-closure.

For $i \geq 1$ set

$$\mathcal{A}_{V,i} = E_{[1,i-1]} \circ \mathrm{Mat}_{V^i} = \sum_{(v_1,\ldots,v_{i-1}) \in V^{i-1}} I_{v_1,\ldots,v_{i-1}} \otimes \mathrm{Mat}_V$$

where $I_{v_1,\ldots,v_{i-1}} = I_{v_1} \otimes \cdots \otimes I_{v_{i-1}}$. Clearly, $\mathcal{A}_{V,i}$ is a subalgebra of Mat_{V^i} closed under the Hadamard multiplication and the Hermitian conjugation. Let us define a linear map

$$\pi_i : \mathcal{A}_{V,i+1} \to \mathcal{A}_{V,i}, \quad i \geq 1,$$

by

$$\pi_i\Big(\sum_{(v_1,\ldots,v_i)} I_{v_1,\ldots,v_i} \otimes A_{v_1,\ldots,v_i}\Big) = \sum_{(v_1,\ldots,v_{i-1})} I_{v_1,\ldots,v_{i-1}} \otimes \sum_{v_i \in V} A_{v_1,\ldots,v_i}. \tag{10}$$

In these terms the procedure from [6] can be described as follows.

Procedure B_m $(m \geq 1)$

Input: a cellular algebra W on V.
Output: a cellular algebra $B_m(W) \geq W$.
Step 1. Construct the set

$$\mathcal{R}_m = \{\widehat{R} \mid R \in \mathcal{R}(W)\} \subset \mathcal{A}_{V,m}$$

and the cellular algebra

$$W(m) = [\mathcal{R}_m, D_{1,m}, \ldots, D_{m-1,m}]$$

where $\widehat{R}$ and $D_{i,m}$ are as in (8) and (5) respectively.
Step 2. For $i = m-1, \ldots, 1$ find sucsessively a linear space

$$W(i) = \pi_i(W(i+1)) \subset \mathcal{A}_{V,i}.$$

Set $W' = [W(1)]$.
Step 3. If $W' \neq W$, then $W := W'$ and go to Step 1. Otherwise, set $B_m(W) = W'$.∎

Proposition 4.2 *Let W be a cellular algebra on V and $m \geq 1$. Then $B_m(W) \leq W^{(m)}$.*

Proof. For $i \in [1, m]$ set

$$W_i = E_{[1,i-1]} \circ (D_{[i,m]} \widehat{W}^{(m)} D_{[i,m]}).$$

Then $W_i \subset h_i(\mathcal{A}_{V,i})$ where $h_i : \mathcal{A}_{V,i} \to \mathcal{A}_{V,m}$ is a linear map induced by the injection $\xi_i^m : V^i \to V^m$ defined in (9). We will prove that

$$\pi'_i(W_{i+1}) \subset W_i \quad \text{for all} \quad i \in [1, m-1]$$

where $\pi'_i = h_i \pi_i h_{i+1}^{-1}$ and π_i is defined by (10).

A straightforward checking shows that

$$\pi'_i(A) = D_{[i,m]} E_{[1,m] \setminus \{i\}} A E_{[1,m] \setminus \{i\}} D_{[i,m]}, \qquad A \in h_{i+1}(\mathcal{A}_{V,i+1}).$$

So $\pi'_i(W_{i+1}) \subset W_i$ for all i. By the definition of $W(m)$ at Step 1 $W(m) \subset W_m$. Therefore,

$$W(1) = \pi_1 \cdots \pi_{m-1}(W(m)) \subset h_1^{-1} \pi'_1 \cdots \pi'_{m-1}(W_m) \subset h_1^{-1}(W_1) = W^{(m)},$$

which completes the proof.■

5 Proof of Theorem 1.3

In this section we prove Theorem 1.3. Given W with $s(W) \leq m-1$ we will show that the algebra $W^{(m)}$ is Schurian.

By the hypothesis of the theorem $W_{v_1,\ldots,v_{m-1}} = \mathrm{Mat}_V$ for some $(v_1, \ldots, v_{m-1}) \in V^{m-1}$. Denote by E the indecomposable component (in $\widehat{W} = \widehat{W}^{(m)}$) of the equivalence $E_{[1,m-1]}$ for which $U = U_{v_1,\ldots,v_{m-1}}$ is one of its classes. By statement (3) of Lemma 3.1 we have $\widehat{W}_{E,U} \geq (W_{v_1,\ldots,v_{m-1}})^{\zeta_U} = \mathrm{Mat}_U$, whence $\widehat{W}_{E,U} = \mathrm{Mat}_U$. By statement (1) of Lemma 2.5 and statement (2) of Lemma 2.4

$$\widehat{W}_{E,U'} = \mathrm{Mat}_{U'} \quad \text{for all} \quad U' \in V^m/E. \tag{11}$$

Statement (2) of Lemma 2.5 implies that

$$\mathrm{supp}(E) = \bigcup_{i=1}^{s} U_i \tag{12}$$

where $U_i \in \mathrm{Cel}(\widehat{W})$ with $U' \cap U_i \neq \emptyset$ for all $U' \in V^m/E$. It follows from (11) that $|U' \cap U_i| = 1$ for all U' and i. In particular, $s = n$.

For any $U' \in V^m/E$ let $f_{U,U'} : \widehat{W}_{E,U} \to \widehat{W}_{E,U'}$ be the weak isomorphism from statement (1) of Lemma 2.5 (with $\widehat{W}$ instead of W). By (11) $f_{U,U'}$ is induced by a bijection $\varphi_{U,U'} : U \to U'$, i.e. $f_{U,U'}(A) = A^{\varphi_{U,U'}}$ for all $A \in \widehat{W}_{E,U}$. Set

$$\theta_{U'} = \zeta_U \varphi_{U,U'} \zeta_{U'}^{-1}, \quad U' \in V^m/E \tag{13}$$

Clearly, $\theta_{U'} \in \mathrm{Sym}(V)$ for all U'. Moreover, by Lemma 3.1 and the definition of the isomorphism $f_{U,U'}$ we have

$$A^{\theta_{U'}} = A^{\zeta_U \varphi_{U,U'} \zeta_{U'}^{-1}} = (I_U \widehat{A} I_U)^{\varphi_{U,U'} \zeta_{U'}^{-1}} = (f_{U,U'}(I_U \widehat{A} I_U))^{\zeta_{U'}^{-1}} = (I_{U'} \widehat{A} I_{U'})^{\zeta_{U'}^{-1}} = A$$

for all $A \in W$ where $\widehat{A} = I_V \otimes \cdots \otimes I_V \otimes A$ (see (8)). Thus

$$\theta_{U'} \in \mathrm{Aut}(W) \quad \text{for all} \quad U' \in V^m/E. \tag{14}$$

We are to show that given $R \in \mathcal{R}(W^{(m)})$ and $(u,v),(u',v') \in R$ there exists $U' \in V^m/E$ such that

$$(u^{\theta_{U'}}, v^{\theta_{U'}}) = (u',v'). \tag{15}$$

Then it will imply by (14) that $\mathrm{Aut}(W)$ acts transitively on each basis relation of $W^{(m)}$, i.e. the cellular algebra $W^{(m)}$ is Schurian.

Let $R \in \mathcal{R}(W^{(m)})$ and $(u,v),(u',v') \in R$. Consider the following path

$$(u,\ldots,u,v) \to (v_1,\ldots,v_{m-1},u) \to (v_1,\ldots,v_{m-1},v) \to (u,\ldots,u,v).$$

Denote its type by (R_0,R_1,R_2) where $R_i \in \mathcal{R}(\widehat{W})$, $i=0,1,2$. Clearly (see (4),

$$R_0 \subset P_{\{(m-1,m)\}}, \quad R_1 \subset E, \quad R_2 \subset P_{\{(m,m)\}}. \tag{16}$$

By statement (1) of Proposition 3.6 the points $(u,\ldots,u,v)$ and $(u',\ldots,u',v')$ belong to the same cell of $\widehat{W}$. So by the Path Proposition there exists a path from $(u',\ldots,u',v')$ to itself of the type (R_0,R_1,R_2). By (16) it is of the form

$$(u',\ldots,u',v') \to (v'_1,\ldots,v'_{m-1},u') \to (v'_1,\ldots,v'_{m-1},v') \to (u',\ldots,u',v')$$

for some $(v'_1,\ldots,v'_{m-1}) \in V^{m-1}$, and $U' = U_{v'_1,\ldots,v'_{m-1}}$ is a class modulo E. To complete the proof it suffices to check that $u^{\theta_{U'}} = u'$ and $v^{\theta_{U'}} = v'$. We prove only the first equality, since the second one is similar.

Since $R_1 \in \mathcal{R}(\widehat{W})$, the points $(v_1,\ldots,v_{m-1},u)$ and $(v'_1,\ldots,v'_{m-1},u')$ belong to the same cell of $\widehat{W}$. From $R_1 \subset E$ it follows that the cell coincides with U_i for some i. Since $|U \cap U_i| = |U' \cap U_i| = 1$ (see above) we have

$$U \cap U_i = \{(v_1,\ldots,v_{m-1},u)\}, \quad U' \cap U_i = \{(v'_1,\ldots,v'_{m-1},u')\}.$$

By the definition of $\varphi_{U,U'}$ (see also Lemma 2.5) we see that $(U \cap U_i)^{\varphi_{U,U'}} = U' \cap U_i$. So

$$(v_1,\ldots,v_{m-1},u)^{\varphi_{U,U'}} = (v'_1,\ldots,v'_{m-1},u').$$

On the other hand, by the definition of $\theta_{U'}$ (see (13))

$$(v_1,\ldots,v_{m-1},w)^{\varphi_{U,U'}} = (v'_1,\ldots,v'_{m-1},w^{\theta_{U'}})$$

for all $w \in V$. Therefore $u^{\theta_{U'}} = u'$. Theorem is proved.■

6 Concluding remarks and Open problems

There is a lot of problems concerning Schurian Polynomial Approximation Schemes. We concentrate here only on two of them.

1. Let $S : W \mapsto S_m(W)$ and $T : W \mapsto T_m(W)$ $(m = 1, 2 \ldots)$ be two Schurian Polynomial Approximation Schemes. We say that S is reducible to T if there exists a function $f: \mathbf{N} \to \mathbf{N}$ where $\mathbf{N} = \{1, 2, \ldots\}$ such that $f(m) \leq cm$ for all $m \in \mathbf{N}$ with some constant $c = c(S, T)$, and $S_m(W) \leq T_{f(m)}(W)$ for all cellular algebras W and all m. S and T are called equivalent if each of them is reducible to the other. Theorem 1.2 shows that the schemes A_m and B_m (see section 4) are reducible to the scheme defined by the m-closure operators.

Problem 6.1 *Are all the three schemes are equivalent?*

We don't know the answer to this question.

2. The algorithmic base of the Schurian Polynomial Approximation Scheme defined by the m-closure operators is the construction of the cellular closure of a set of matrices. This problem can efficiently (in polynomial time) be solved by the standard Weisfeiler-Lehman algorithm.

Problem 6.2 *Is the above problem in* $\mathbf{NC}$*? In other words, can the cellular closure of a* $n \times n$*-matrix be found by* $n^{O(1)}$ *parallel computers in time* $(\log n)^{O(1)}$*?*

(For the exact definition of **NC** and related concepts see [8].) The main difficulty here is that the cellular closure is defined by means of two binary operations (the ordinary matrix multiplication and the Hadamard one) which don't commute with each other. Note, that for each of them the problem of constructing the closure with respect to it is in **NC**.

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