Lower Bound for Randomized Linear Decision Tree Recognizing a Union of Hyperplanes in a Generic Position

Dima Grigoriev* **Marek Karpinski†**

Abstract

Let L be a union of hyperplanes with s vertices. We prove that the runtime of a probabilistic linear search tree recognizing membership to L is at least $\Omega(\log s)$, provided that L satisfies a certain condition which could be treated as a generic position. A more general statement, namely without the condition, was claimed by F. Meyer auf der Heide [1], but the proof contained a mistake.

*Departments of Computer Science and Mathematics, Penn State University, University Park, PA 16802. Email: dima@cse.psu.edu.

†Department of Computer Science, University of Bonn, 53117 Bonn, and International Computer Science Institute, Berkeley, California, Email: marek@cs.uni-bonn.de.

1 Families of hyperplanes in a generic position

Let $L = \bigcup_{1 \le i \le m} H_i \subset \mathbb{R}^{\kappa}$ be a union of hyperplanes. We intend to define a version of what does it mean that L is in a generic position.

If $Q = \bigcap_{1 \le j \le t} H_{i_j}$ has the dimension $\dim Q = l$ we call Q l–face of L. Also 0–faces we call vertices. If a hyperplane H contains some l–face for rather l then H contains many vertices of L. The generic position for L means, informally speaking, that this is the only reason for H to contain many vertices of L.

Definition. We say that L is in a generic position if for some $c_1 > c_2 > 0, c_3 > 0$ and any hyperplane $H \subset \mathbb{R}^{\kappa}$

1) L has $s \ge m^{c_1 n}$ vertices,

2) each vertex belongs to exactly n hyperplanes of L,

3) the number of vertices v lying in H for which there is no l–face contained in H such that this l–face contains v, where $l \ge c_3 n$, does not exceed $m^{c_2 n}$.

One can show that if $H_1, \ldots, H_m$satisfy the property of algebraically independence, namely, that $m \cdot n$ coefficients a_{ij} of all linear equations for $H_1, \ldots, H_m$ (i.e. $H_i = \left\{ \sum_{1 \le j \le n} a_{ij} X_j = 1 \right\}$) are algebraically independent over Q then L is in a generic position.

Moreover, one can prove in this case the following. Let $Q_1, \ldots, Q_t$ be all maximal (in the sense of inclusion) faces of L containes in H, then $\sum_{1 \le i \le t} \dim(Q_i + 1) \le n$. Thus, the number of vertices in the item considered in the item 3 of the definition does not exceed $n \cdot m^{c_3 n}$ since any l–face cannot contain more than m^l vertices of L.

Let D be a probabilistic linear search algorithm (or briefly α–*PLSA*) recognizing L with two–sided error $\alpha < 1/2$ (one can find in [1], [2] the concepts used in the present paper).

Theorem. If L is in a generic position then the runtime of D is greater than $\Omega\left(n \log m\right)$.

Note that the similar result was claimed in [1] even without the condition 3) from the definition of a generic position, but the proof contained a mistake.

For a value of the random parameter $0 \le \gamma \le 1$ by D_γ we denote the corresponding *LSA* (cf. [1]).

Recall that in [2] it is proved that one can obtain β–*PLSA* recognizing the same language L as D for any cinstant $\beta > 0$ increasing the runtime of D by at most a constant factor. We shall use this remark to make α as small as desired.

As in [1] one shows that for any vertex of L there exists $\epsilon > 0$ such that each hyperplane occuring as a testing one in D which intersects the closed ball $B_\epsilon(v)$ of the radius ϵ and with the center in v, should pass through v.

Similar to [1] select from D all the testing hyperplanes passing through v. Then the obtained thereby D' is α-$PLSA$ recognizing the language $L \cap B_\epsilon(v)$, when being restricted on $B_\epsilon(v)$.

Making a suitable affine transformation, we can assume that v is the coordinate origin and besides, the hyperplanes from L passing through v, are just the coordinate hyperplanes $\{X_1 = 0\}, \ldots, \{X_n = 0\}$.

For any $0 \leq \gamma \leq 1$ each leaf of D'_γ provides a polyhedra V of the form

$$\{L_1 = 0\} \cap \ldots \cap \{L_{q_1} = 0\} \cap \{L_{q_1+1} > 0\} \cap \ldots \cap \{L_q > 0\}$$

for some testing hyperplanes $L_1, \ldots, L_q$. Then $P = \{L_1 = \ldots = L_q = 0\}$ is the minimal (in the sense of inclusion) face of the closure of V. If $q_1 = 0$ then V is open. Polyhedra corresponding to all the leaves of D'_γ form the partition $\mathbb{R}^K$.

For the time being we fix $0 \leq \gamma \leq 1$ and an open polyhedron V. Denote by $\Delta(V)$ the maximal dimension of the faces of L passing through v which are contained in P. Any such face of L has the form $\bigcap_{i \in I}\{X_i = 0\}$ for a certain subset $I \subset \{1, \ldots, n\}$. Observe that if two faces $\bigcap_{i \in I}\{X_i = 0\}$ and $\bigcap_{i \in J}\{X_i = 0\}$ of L are contained in P then the face $\bigcap_{i \in I \cap J}\{X_i = 0\}$ is contained in P as well. Thus, there is the unique maximal face of the form $\bigcap_{i \in I}\{X_i = 0\}$ contained in P and its dimension equals to $\Delta(V)$.

2 Estimating spherical measure of intersections of a polyhedron with the coordinate hyperplanes

For any set $W \subset \mathbb{R}^K$ consider its cone $C(W)$ with the vertex in the rorigin and by $\delta_n(W) = \mu_n(C(W) \cap B_1)/\mu_n(B_1)$ where μ_n is the usual Borel measure in $\mathbb{R}^K$ and the ball $B_1 = B_1(0)$ (we consider only measurable sets).

Take any line $h \in P$ passing through the origin (provided that $\dim P > 0$ and such a line does exist) and let H be a hyperplane orthogonal to h and passing through the origin.

Lemma 1. $\delta_n(V) = \delta_{n-1}(V \cap H)$

Proof. Actually, a more general statement holds. For any subset $U \subset H$ for the direct product $U \times h \subset \mathbb{R}^K$ we have $\delta_n(U \times h) = \delta_{n-1}(U)$. To prove the latter statement one can consider a partition of $H \cap B_1 = U_0 \cup \ldots \cup U_t$ into "small" pieces where $U_i = \mathcal{R}_i(U_0)$, $1 \leq i \leq t$ for appropriate rotations $\mathcal{R}_i$ of H. Extend every $\mathcal{R}_i$ to the rotation $\overline{\mathcal{R}_i}$ of $\mathbb{R}^K$ by leaving h invariant. Then $1 = \delta_n(B_1) = (t+1)\delta_n(U_0 \times h)$ and $1 = \delta_{n-1}(H \cap B_1) = (t+1)\delta_n(U_0)$. The standard arguing with approximation of U by a partitioning into "small" pieces completes the proof of the lemma.

□

Lemma 2. *If α–PLSA D' recognizes the language $L \cap B\epsilon(v)$ (where L is in a generic position), being restricted on $B_\epsilon(v)$, where v is a vertex of L, then with a probability $\geq p = 1 - \frac{2\alpha}{c_3}$ (thus, we assume that $\alpha < {}^{c_3}/_2$, see the remark in section 1), a certain leaf of D'_γ provides an open polyhedron V with $\Delta(V) \leq c_3 n$.*

Proof. Suppose the contrary. Recall that we assume that v coincides with the origin and among the hyperplanes $H_1, \dots, H_m$ there are $\{X_1 = 0\}, \dots, \{X_n = 0\}$. Then $1 = \sum \delta_n(V)$ where the summation ranges over all open polyhedra V provided by the leaves of D'_γ. Assume that for a particular value of the random parameter $0 \leq \gamma \leq 1$ for all open V we have $\Delta(V) > c_3 n$.

Let P be the minimal face of V, then $P \subset \{X_{i_1} = \dots = X_{i_l} = 0\}$ for some indices $1 \leq i_1, \dots, i_l \leq n$ with $l < (1-c_3)n$. For any index $j \in \{1, \dots, n\} \setminus \{i_1, \dots, i_l\}$ lemma 1 entails $\delta_{n-1}\big(V \cap \{X_j = 0\}\big) = \delta_n(V)$. Therefore $\sum_V \sum_{1 \leq i \leq n} \delta_{n-1}\big(V \cap \{X_i = 0\}\big) > c_3 n$. By the supposition the expectation of the latter sum over the values of the random parameter $0 \leq \gamma \leq 1$ is greater than

$$E\Big(\sum_V \sum_{1 \leq i \leq n} \delta_{n-1}\big(V \cap \{X_i = 0\}\big)\Big) > (1-p)c_3 n = 2\alpha n.$$

This contradicts to the definition of α–PLSA taking into account that for any point from $V \cap \{X_i = 0\}$ the output of D'_γ is the same as for the points, from its small neighbourhood, so D'_γ does not distinguish them. The obtained contradiction proves the lemma.

□

3 Lower bound on the number of faces in *PLSA*

Now we complete the proof of the theorem, the arguing is similar to one in [1]. Applying lemma 2 to each vertex of L we conclude that there exists a value $0 \leq \gamma \leq 1$ of the random parameters such that for at least ps vertices v of L there is an open polyhedron V provided by corresponding to a leaf of D_γ such that V hast a face P (which could be not a minimal face of P unlike the local situation in section 2) containing v and if some l–face of L is contained in P and contains v then $l \leq c_3 n$. To every such vertex v let us correspond a face p (if there are several such faces then correspond any of them).

Since L is in a generic position (see the definition), any face P of D_γ could be corresponded to at most $m^{c_2 n}$ vertices of L. Hence there are at least ${}^{ps}/_{m^{c_2 n}} = pm^{(c_1 - c_2)n}$ faces of D_γ. But on the other hand, the number of faces in D_γ does not exceed s^{2T} (cf. [1]), therefore $2^{2T} \geq pm^{(c_1-c_2)n}$, this completes the proof of the theorem.

References

[1] Meyer auf der Heide, F., *Nondeterministic Versus Probabilistic Linear Search Algorithms*, in *Proc. IEEE Symp. Found. of Comput. Sci. (1985)*, pp. 65–73.

[2] Meyer auf der Heide, F., *Simulating Probabilistic by Deterministic Algebraic Computation Trees*, in *Theor. Comput. Sci.* $\underline{41}$ *(1985)*, pp. 325–330.