Lower Bounds on Complexity of Testing Membership to a Polygon for Algebraic and Randomized Computation Trees

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Abstract

We describe a new method for proving lower bounds for algebraic computation trees. We prove, for the first time, that the minimum depth for arbitrary computation trees for the problem of testing the membership to a polygon with N nodes is $\Omega(\log N)$. Moreover, we prove that the corresponding lower bound for the randomized computation trees matches the above bound. Finally, we prove that for the algebraic exp-log computation trees (cf. [GSY 93]), the minimum depth is $\Omega(\sqrt{\log N})$. We generalize the last result to the multidimensional case, showing that if an exp-log computation tree tests a membership to a semialgebraic set with a sum of Betti numbers M, then the depth of a tree is at least $\Omega(\sqrt{\log M})$.

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Introduction

A problem of testing membership to a semialgebraic set was considered by many authors see, e.g. [B 83], [BLY 92], [BL 92], [Y 93], [MH 84], [MH 85], [YR 80] and the references there). Here we consider a problem of testing membership to a polygon (with N nodes). In [MH 84] it was shown, in particular, that for this problem $\mathcal{O}(\log N)$ upper bound is valid for linear decision trees, in [YR 80] it was shown to be a lower bound.The similar question was open for the algebraic computation trees, in this paper we prove a lower bound $\Omega(\log N)$ for algebraic computation trees (corollary 1) and for the randomized algebraic computation trees (corollary 2).

The main tool for proving complexity lower bounds for algebraic computation trees was the number of connected components of a tested set or of its complement, introduced in [B 83], and this invariant was generalized first to the Euler characteristic ([BLY 92]), then to the sum of Betti numbers in [Y 93]. Also in [BLY 92] another method for linear decision trees was designed based on the minimal number of convex sets necessary to compose a given one. All the mentioned powerful tools unfortunately fail for algebraic trees in the simple case of a (convex) polygon. To handle it, we introduce another method based on a number of singular points of a boundary (being a curve) of a tested semi–algebraic set on the plane (proposition 1). This approach hopes to be prospective, but requires to overcome several difficulties in the multidimensional situation which seem to be tractable.

Another way to extend the class of the algebraic trees is to add other gate functions in addition to the presented arithmetic ones. An interesting example (which was already considered in [GSY 93]), is to add exp– and log–gates. In [GSY 93] it was shown that adding exp and log does not improve the complexity of computing an algebraic function (under some natural conditions). It is an open question, whether adding exp and log increases the power of algebraic computation trees. We make a step towards it (proposition 3) proving a lower bound, namely the square root of logarithm of the number of singular points. In order to improve it getting rid of the square root, one should apparently improve an upper bound on the number of roots in a Pfaffian chain ([K 91]) diminishing a gap with lower bounds provided by the known examples, which seems to be unlikely at the moment. Similarly we obtain a lower bound $\sqrt{\log N}$ for testing membership to the polygon with N nodes by an exp–log–tree (corollary 3).

At the end we observe (proposition 3) that the known bounds (in the multidimensional case) from [B 83], [Y 93] are valid for exp–log–trees again with weakening them by taking a square root.

1 A lower bound on the depth of algebraic trees based on the number of singular points

Let $S \subset \mathbb{R}^2$ be a closed semialgebraic set on a plane and let T be an algebraic computation tree (see e.g. [B 83], [Y 93], [MH 84]), which tests a membership problem for S whose boundary ∂S is a semialgebraic curve. Later we call T simply an algebraic tree. Let $0 \neq f \in \mathbb{R}[X_1, X_2]$ be a polynomial. For any point $v = (v_1, v_2)$ of the semialgebraic curve $V_f = \{(x_1, x_2) \in \mathbb{R}^2 : f(x_1, x_2) = 0\}$ define its multiplicity $m(v) = m_f(v)$ as the maximal m such that the partial derivates

$$\frac{\partial^{i_1+i_2} f}{\partial X_1^{i_1} \partial X_2^{i_2}}(v) = 0$$

vanish for all $i_1 + i_2 < m$. A point $v \in V_f$ is singular if either v is a connected component of V_f (i.e. an isolated point) or the multiplicity $m(v)$ is larger than the multiplicities of points of V in a certain neighborhood of v on V_f (c.f. [GKS 93]). Observe that an isolated point could be treated as singular by the above definition as for the points out of V_f their multiplicities equal to zero. Evidently, if at least two branches of a curve pass through a point, then this point is a singular (surely, the inverse is not true).

Consider a branch B of the tree T and let u_1, u_2 be auxiliary variables along B (c.f. [B 83]) where for each $1 \leq j \leq k$, $u_j = w_1 \odot w_2$, here $w_i, i = 1, 2$ is either X_1, either X_2, either a real number, or u_p for some $1 \leq p < j$, and $\odot$ is either "+" or "×". Every u_j could be developped (by induction on j) in a polynomial $g_j \in \mathbb{R}[X_1, X_2]$. Let the conditions along the branch B be $g_1 \epsilon_1 0, \ldots, g_k \epsilon_k 0$ where each ϵ_j is either "=" or ">". Consider a set $U_B = \{g_1 \epsilon_1 0, \ldots, g_k \epsilon_k 0\}$ computed along B and a curve $W_b = \{g_j = 0, \epsilon_j = \text{"="}\} \cup \{\prod g_j = 0, \epsilon_j = \text{">"}\} \subset \mathbb{R}^2$. Obviously $\partial U_B \subset W_B$.

Lemma 1. For any singular point v of the boundary ∂S there exists a branch B of T such that v is either a singular point of W_B, either an isolated point of the set $\{g_j = 0, \epsilon_j = \text{"="}\}$ or an isolated point of the set $\{\prod g_j = 0, \epsilon = \text{">"}\}$.

Proof. Assume the contrary. Consider all the branches $B_1, \ldots, B_s$ of T such that $v \in \overline{U_{B_1}}, \ldots, v \in \overline{U_{B_s}}$ (here the bar denotes the closure in the euclidian topology). Since $v \in \partial U_{B_1}, \ldots, v \in \partial U_{B_s}$, we conclude that $v \in W_{B_1} \cap \ldots \cap W_{B_s}$. There is exactly one set among $U_{B_1}, \ldots U_{B_s}$ (let it be U_{B_1} for definiteness), for which $v \in U_{B_1}$. Fix any $2 \leq j \leq s$, let $h_1, \ldots, h_l \in \mathbb{R}[X_1, X_2]$ be polynomials along the branch B_j and let the conditions along B_j be $h_1 = 0, \ldots, h_p = 0, h_{p+1} > 0, \ldots, h_l > 0$. If $p > 0$, then v cannot belong to a curve $\{h_1 = \ldots = h_p = 0\}$ (unless, v is an isolated point either of the latter set or of the set $\{\prod_{p+1 \leq j \leq l} h_j = 0\}$, in both cases the statement of the lemma is valid), as otherwise $v \in \{h_{p+1} \cdots h_l = 0\}$ (taking into account that $v \notin U_{B_j}$) and v appears to be a singular point of the curve W_{B_j} (unless, the curves $\{h_1 = \ldots = h_p = 0\}$ and $\{h_{p+1} \cdots h_l = 0\}$ coincide in a small enough neighborhod of the point v on $\mathbb{R}^2$), which contradicts to the assumption.

Let the conditions along B_1 be $h_1^{(0)} = \ldots = h_q^{(0)} = 0, h_{q+1}^{(0)} > 0, \ldots h_l^{(0)} > 0$. Observe that the boundary ∂S is contained in the union of the curves of the form $\{h_1 = \ldots = h_p = 0\}$ for different branches, therefore the proved above implies that in a small enough neighborhood of v the boundary ∂S coincides with the intersection of the curve $C = \{h_1^{(0)} = \ldots h_q^{(0)} = 0\}$ with this neighborhood. Evidently $v \notin \{h_{q+1}^{(0)} \cdots h_l^{(0)} = 0\}$. Hence v should be a singular point of the curve C, which contradicts to the assumption and proves the lemma.

Now we proceed to estimating the number of singular points of a curve W_B as well as the numbers of isolated points of the sets $\{g_j = 0, \epsilon = `='\}$ and $\{\prod g_j = 0, \epsilon_j = `>'\}$ (see lemma 1).

Denote $G = (\sum_{\epsilon_j = `='} g_j^2) \prod_{\epsilon_j = `>'} g_j$. For any $m \geq 0$ consider a system $\mathcal{S}_m$ of equations to zero of all the partial derivates of G up to the order m. Denote by s_m the number of connected components of a semialgebraic set determined by $\mathcal{S}_m$. As $\deg g_i \leq 2^i$, the number of singular points of W_B does not exceed $\sum_{0 \leq m < 2^{k+1}} s_m \leq 2^{\mathcal{O}(k)}$ (see [M 64]). The similar bounds are valid also for the mentioned above numbers of isolated points.

Therefore, if all the branches in T have a length at most c, then by lemma 1 and just obtained bounds we conclude that the number of singular points of ∂S does not exceed $2^{\mathcal{O}(c)}$. Thus, we get the following proposition.

PROPOSITION 1. The depth of an algebraic tree testing the membership problem to a closed semialgebraic set $S \in \mathbb{R}^{\nvDash}$ can be bounded from below by the logarithm of the number of singular points of ∂S (up to a constant factor).

Remark that it is not necessary to require closedness of S.

COROLLARY 1. The depth of an algebraic tree testing membership problem to a polygon with N nodes, is at least $\Omega(\log N)$.

2 Randomized algebraic trees

Consider now a randomized algebraic tree $\mathcal{T}$ (for the definitions see [MH 85]) testing membership problem to $S \in \mathbb{R}^{\nvDash}$, say, with a probability $^2/_3$. Take any singular point v of ∂S. Assume first that v is not isolated. Let $C_1, \ldots, C_q$ be all the curves of ∂S in a small enough neighborhood of v, passing through v and being algebraically irreducible over $\mathbb{R}$. It is not difficult to see that each of C_i, $1 \leq i \leq q$ should be contained in the boundary of a tested set with a probability at least $^2/_3$. So, if $q = 1$ then v is a singular point of C_1, and v appears as a singular point of a boundary of a tested set with a probability at least $^2/_3$. If $q \geq 2$ then v appears as an intersection of at least two curves among C_i, $1 \leq i \leq q$ with a probability at least $^1/_3$, again in this case v is a singular point of a boundary of a tested set.

If v is an isolated point of ∂S then it appears as an isolated point of a tested set with a probability at least $^2/_3$.

Hence there exists a value of a random parameter for which at least $^1/_3$ of all the singular points of ∂S (including isolated points) would appear as singular points of a boundary of a set tested by the tree corresponded to the mentioned value of the random parameter. Now we apply the proposition 1 to this tree and get the following proposition.

PROPOSITION 2. The depth of a randomized algebraic tree testing the membership problem to a closed semialgebraic set $S \in \mathbb{R}^{\nvDash}$ can be bounded from below by the logarithm of the number of singular points of ∂S (up to a constant factor).

COROLLARY 2. The depth of a randomized algebraic tree testing membership problem to a polygon with N nodes, is at least $\Omega(\log N)$.

3 Trees with exp and log

Now we consider an extension of the class of algebraic trees, allowing also in the definition of the tree two extra operations (c.f. above): $u_j = \exp w$ or $u_j = \log w$, where w is either X_1, either X_2 or $w = u_p$ for some $1 \leq p < j$. Let us call them exp–log–trees.

The proof of lemma 1 goes through literally for exp–log–trees. To estimate the number of singular points of W_B apply [K 91] using that a chain of the functions $g_1, \ldots, g_k$ computed along B, is a particular case of a Pfaffian chain introduced in [K 91], and thereby the number of singular points of W_B does not exceed $2^{\mathcal{O}(k^2)}$. Finally, from the mentioned modification of lemma 1 we get the following proposition.

PROPOSITION 3. The depth of an exp–log–tree testing the membership problem to a closed semialgebraic set $S \in \mathbb{R}^{\not{\mathbb{C}}}$ can be bounded from below by the square root of the logarithm of the number of singular points of ∂S (up to a constant factor).

COROLLARY 3. The depth of an exp–log–tree testing membership problem to a polygon with N nodes, is at least $\Omega(\sqrt{\log N})$.

Notice that the proposition 3 is valid as well for the randomized exp–log–trees, the proof goes as in the section 2 with the only diference that one should consider analytical irreducible components rather than algebraic ones. Observe also that the bounds from [B 83], [Y 93] could be extended (with weakening by taking a square root as in the proposition 3) to exp–log–trees, following the proofs in [B 83], [Y 93] and using for bounding the sum of Betti numbers [K 91] instead of [M 64] (which is usually being invoked in the algebraic case). We would like to stress that in the next proposition we deal with a multidimensional case unlike all the previous contents of the paper.

PROPOSITION 4. If an exp–log–tree tests a membership to a semialgebraic set with a sum of Betti numbers M, then the depth of the tree is at least $\Omega(\sqrt{\log M})$.

Let us notice that an example with the presentation of n–degree Chebyshev polynomial (having n real roots) $\cos(n \arccos x)$ shows that one should be careful in extending the set of allowed gate functions for the trees (thus, if we would add cos and arccos as the gate functions, the obtained above bounds would not be valid anymore).

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