Subclasses of Quantified Boolean Formulas

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Abstract

Using the results of a former paper of two of the authors, for certain subclasses of quantified Boolean formulas it is shown, that the evalution problem is coNP-complete. These subclasses can be seen as extensions of Horn and 2-CNF formulas.
Further it is shown that the evalution problem for quantified Boolean formulas remains PSPACE-complete, even if at most one universal variable is allowed in each clause.

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1 Introduction

It is well known that the evaluation problem for Boolean formulas, that means to determine wether a formula $Q_1z_1 \ldots Q_nz_n(\alpha_1 \wedge \ldots \wedge \alpha_n)$ is true, where Q is either $\exists$ or $\forall$ and α_i is a propositional clause, is PSPACE-complete [StM 73]. There are also subclasses decidable in polynominal time. In [APT 78] a linear time algorithm for quantified Boolean formulas with clauses of at most 2 variables only is given. In [KKBS 87] a cubic time algorithm for quantified Horn formulas was given. This result was improved in [KaKB 90]. It was shown that the evaluation problem for quantified Horn formulas can be decided in $O(rn)$ time, where n is the length of the formula and r is the number of universal variables occuring positive in the formula. In the same paper a complete and sound resolution operation for quantified Boolean formulas, called Q-Resolution, was introduced. A Q-Resolution is an ordinary resolution, where only existential variables can be matched and additionally, each universal variable of a clause, for which no existential variable exists in that clause occuring after the universal variable in the prefix, is omitted.

A natural extension of quantified Horn formulas, where universal literals of a clause can be arbitrary, but the existential literals are in Horn form, was introduced.
For this subclass, called extended quantified Horn formulas, it was shown by means of Q-Resolution, that the evaluation problem for such formulas with prefix $\forall \ldots \forall \ldots \exists \ldots \exists$ is coNP-complete. In this paper our interest is the complexity of extended quantified Horn formulas and extended quantified 2-CNF formulas, that are formulas with at most two existential variables and an arbitrary number of universal variables in each clause.

It is shown that for any fixed number of occurences of the pattern $\forall \ldots \forall \ldots \exists \ldots \exists \ldots$ in the prefix the evalution problem for extended quantified Horn formulas remains coNP-complete.

A similar result is shown for extended quantified 2-CNF formulas.

As mentioned above the evaluation problem for quantified Boolean formulas is PSPACE-complete. In the last part of this paper it is shown, that this is true, even if at most one universal variable in each clause exists. So there is no improvement of the complexity possible by similar restrictions of the universal variables of a formula.

2 Preliminaries

In what follows quantified Boolean formulas are of the form

$$\begin{aligned} \Phi &= \forall x_1 \exists y_1 \ldots \forall x_k \exists y_k (\phi_1 \wedge \ldots \wedge \phi_m), && \text{where} \\ x_i &= \mathrm{x}_{n_{i-1}+1} \ldots \mathrm{x}_{n_i} && \text{and } n_0 = 1, \\ y_i &= \mathrm{y}_{m_{i-1}+1} \ldots \mathrm{y}_{m_i} && \text{and } m_0 = 1. \end{aligned}$$

Further we assume that the variables in a clauses are given in a fixed order. We say a literal L_1 is before a literal L_2, if the variable of L_1 occurs in the order of the prefix before the variable of L_2. We write clauses in the form $(L_1 \vee \ldots \vee L_t)$, where L_{i-1} is before L_i. An *X-literal* resp. Y-literal is a literal of the form x_e or $\bar{\mathrm{x}}_e$ resp. y_e or $\bar{\mathrm{y}}_e$. A pure *X-clause* is a non-tautological clause consisting of X-literals only. In particular the empty clause is a pure X-clause.
Finally a formula does not contain tautological clauses and each variable of the prefix occurs in some clause.

Definition (1) *A formula* $\Phi = \forall x_1 \exists y_1 \ldots \forall x_k \exists y_k (\phi_1 \wedge \ldots \wedge \phi_r)$ *is an* extended quantified *Horn Formula resp. 2-CNF formula, if for each* ϕ_i *the Y-part is a Horn-clause resp. 2-CNF-clause, i.e. the clause* ϕ_i *omitting all X-literals is a Horn-clause resp. 2-CNF-clause.*

In the following a prefix $\forall \ldots \forall \exists \ldots \exists$ is denoted as Π_1 and Π_{k+1} is defined as $\forall \ldots \forall \exists \ldots \exists \Pi_k$

Definition (2) *For each* $k \geq 1$ *the class of extended quantified Horn resp. 2-CNF formulas with prefix in* Π_k *is denoted as* Π_k*(extended Horn) resp.* Π_k*(extended 2-CNF).*

Now we introduce our generalized resolution operation, called Q-resolution:

Definition (3)

a: *Let* α *be a non-tautological X-clause, then we replace* α *by the empty clause. In each other non-tautological clause we delete all X-literals, that are not before any Y-literal.*

b: *Let* α_1 *be a clause with Y-literal* y_l *and* α_2 *be a clause with Y-literal* $\overline{\mathrm{y}_l}$*, then the resolvent* α *of* α_1 *and* α_2 *is obtained as follows:*

1. *Remove all occurences of* y_l *and* $\overline{\mathrm{y}_l}$ *in* $\alpha_1 \vee \alpha_2$*.*
2. *Remove all occurences of X-literals, that are not before any Y-literal occuring in the disjunction.*
3. *If the resulting clause contains complementary literals, then no resolvent exists. Otherwise the resulting clause is the resolvent.*

Comparing ordinary resolution with Q-resolution, we see that only literals bounded by existential quantifiers can be matched and universal variables not before an Y-literal will be eliminated.

Example:

$$\forall \mathrm{x}_1\mathrm{x}_2 \exists \mathrm{y}_1\mathrm{y}_2 \forall \mathrm{x}_3((\mathrm{x}_1 \vee \overline{\mathrm{x}_2} \vee \overline{\mathrm{y}_1} \vee \overline{\mathrm{y}_2} \vee \mathrm{x}_3) \wedge (\overline{\mathrm{x}_1} \vee \overline{\mathrm{y}_1}) \wedge (\overline{\mathrm{x}_2} \vee \mathrm{y}_2) \wedge (\overline{\mathrm{x}_1} \vee \overline{\mathrm{y}_2}))$$

The following Q-resolution steps can be performed:

$$(\mathrm{x}_1 \vee \overline{\mathrm{x}_2} \vee \overline{\mathrm{y}_1} \vee \overline{\mathrm{y}_2} \vee \mathrm{x}_3),\ (\overline{\mathrm{x}_2} \vee \mathrm{y}_2) \ \vdash_{\overline{Q-Res}} \ (\mathrm{x}_1 \vee \overline{\mathrm{x}_2} \vee \overline{\mathrm{y}_1})$$

$$(\overline{\mathrm{x}_1} \vee \overline{\mathrm{y}_2}), (\overline{\mathrm{x}_2} \vee \mathrm{y}_2) \ \vdash_{\overline{Q-Res}} \sqcup$$

where $\vdash_{\overline{Q-Res}}$ denotes the application of Q-resolution steps.

It is well known that resolution is complete and sound for propositional formulas in CNF, i.e. a formula $\exists \mathrm{y}_1 \ldots \mathrm{y}_n(\alpha_1 \wedge \ldots \wedge \alpha_r)$ is false iff the empty clause can be derived by resolution applied to $\alpha_1 \wedge \ldots \wedge \alpha_r$.

A similar result for Q-resolution and quantified Boolean formulas is shown in [KaKB 90]:

Theorem 2.1 *A quantified Boolean formula* Φ *is false iff* $\Phi \vdash_{\overline{Q-Res}} \sqcup$

Analogously to the case of unit-resolution for ordinary resolution we can define a Q-unit resolution. A clause ϕ is called a *Y-unit clause* if ϕ contains exactly one Y-literal and arbitrarily many X-literals. A *positive* Y-unit clause is a unit clause with a positive Y-literal.

Definition (4) *The Q-unit-resolution (Q-U-Res) is a Q-resolution, where one of the clauses is a positive Y-unit clause.*

The Q-unit-resolution is useful not only for quantified Horn formulas, but for extended quantified Horn formulas too (see again [KaKB 90]):

Theorem 2.2 *An extended quantified Horn formula* Φ *is false iff* $\Phi \vdash_{\overline{Q-U-Res}} \sqcup$.

3 The Evalution Problem for extended Formulas

Theorem 3.1 *The evaluation problem for* Π_1*(extended Horn) and* Π_1*(extended 2-CNF) is coNP-complete.*

Proof: That the falsity of an extended quantified Horn formula and an extended quantified 2-CNF formula with prefix $\forall\exists$ can be decided nondeterministically in polynominal time is obvious. Thus the evaluation problem belongs to coNP.

Now we associate to each formula $\alpha = \alpha_1 \wedge \ldots \wedge \alpha_n$ in 3-CNF with variables $\mathrm{x}_1, \ldots, \mathrm{x}_m$ a quantified Boolean formula, which is in Π_1(extended Horn) and in Π_1(extended 2-CNF), such that the propositional formula α is satisfiable iff the extended quantified Boolean formula is false.

Let $\mathrm{y}_0, \ldots, \mathrm{y}_n$ be new variables, then we associate to each clause $\alpha_i = (L_{i_1} \vee L_{i_2} \vee L_{i_3})$ three clauses $\phi_{i_1} = (L_{i_1} \vee \overline{\mathrm{y}}_{i-1} \vee \mathrm{y}_i)$, $\phi_{i_2} = (L_{i_2} \vee \overline{\mathrm{y}}_{i-1} \vee \mathrm{y}_i)$ and $\phi_{i_3} = (L_{i_3} \vee \overline{\mathrm{y}}_{i-1} \vee \mathrm{y}_i)$.

Then we define $\Phi(\alpha) := \forall \mathrm{x}_1 \ldots \mathrm{x}_m \exists \mathrm{y}_0 \ldots \mathrm{y}_n\ (\mathrm{y}_0 \wedge \overline{\mathrm{y}}_n \wedge \phi_{1_1} \wedge \phi_{1_2} \wedge \ldots \wedge \phi_{n_3})$. If $\Phi(\alpha)$ is false then it yields $\Phi(\alpha) \vdash_{\overline{Q-Res}} \sqcup$. Since for each $i(1 \leq i \leq n)$ one of the clauses ϕ_{i_1} or ϕ_{i_2} or ϕ_{i_3} must be matched, there is some $L_{i_{j_i}}$ in α_i, such that $\{L_{1_{j_1}}, \ldots, L_{n_{j_n}}\}$ does not contain complementary literals. Then, we can define a satisfying truth assignment $\mathcal{J}$ for α choosing $\mathcal{J}(L_{i_{j_i}}) = 1$.

In the converse direction let α be satisfiable. Then there is some truth assignment $\mathcal{J}$, such that $\mathcal{J}(\alpha) = 1$. Hence, for each i there is some literal $L_{i_{j_i}}$ with $\mathcal{J}(L_{i_{j_i}}) = 1$. Now we can apply the Q-resolution to $\Phi(\alpha)$ obtaining the empty clause as follows:

$(\mathrm{y}_0), (L_{1_{j_1}} \vee \overline{\mathrm{y}}_0 \vee \mathrm{y}_1) \vdash_{\overline{Q-Res}} (L_{1_{j_1}} \vee \mathrm{y}_1), (L_{2_{j_2}} \vee \overline{\mathrm{y}}_1 \vee \mathrm{y}_2) \vdash_{\overline{Q-Res}} (L_{1_{j_1}} \vee L_{2_{j_2}} \vee \mathrm{y}_2), (L_{3_{j_3}} \vee \overline{\mathrm{y}}_2 \vee \mathrm{y}_3) \vdash_{\overline{Q-Res}} \ldots \vdash_{\overline{Q-Res}} (L_{1_{j_1}} \vee \ldots \vee L_{n_{j_n}} \vee \mathrm{y}_n) , \overline{\mathrm{y}}_n \vdash_{\overline{Q-Res}} \sqcup$.

Since the satisfiability problem for 3-CNF is NP-complete and the transformation can be performed in polynominal time, we have proved our desired result.

q.e.d.

For technical reasons we need the following definition:
A quantified Boolean formula $\Phi = \forall x_1 \exists y_1 \ldots \forall x_k \exists y_k (\phi_1 \wedge \ldots \wedge \phi_m)$ is called *reduced*, if each variable in the prefix occurs in $\phi_1 \wedge \ldots \wedge \phi_m$ as literal and for each i the quantified Boolean formula $\Phi - \{\phi_i\}$ is true.

In the case of the falsity of Φ reduced means, that removing one clause we obtain a formula, which is true.

For each quantified Boolean formula $\Phi = \forall x_1 \exists y_1 \ldots \forall x_k \exists y_k (\phi_1 \wedge \ldots \wedge \phi_m)$ a reduced formula Φ^r can be obtained by removing nondeterministically some clauses and removing all variables in the prefix that do not occur in the remaining clauses.
It is easy to see that if Φ is false there is a selection of clauses so that Φ^r is false and reduced and that if Φ is true, Φ^r remains true.

Theorem 3.2 *For each fixed $k \geq 1$ and Φ in Π_k(extended Horn): Φ is false iff $\Phi \vdash_{\overline{Q-U-Res}} \sqcup$ in at most n^k (nondeterministic) Q-unit-resolution steps, where length(Φ) = n.*

Proof: We know that for a formula Φ, which is false, there is a derivation to the empty clause by means of Q-unit-resolution.
We will prove by induction on k that the number of Q-unit-resolution steps can be restricted to n^k.

For $k = 1$ the formula Φ has the form $\forall x_1 \exists y_1 (\phi_1 \wedge \ldots \wedge \phi_m)$. Obviously at most length (Φ) resolution steps are sufficient, because for a propositional Horn formula β length(β) unit-resolution steps lead to the empty clause.

Now let be $k > 1$. We assume $\Phi = \forall x_1 \exists y_1 \ldots \forall x_k \exists y_k (\phi_1 \wedge \ldots \wedge \phi_m)$ is false and reduced. For a clause ϕ and a set S of variables $\phi(S)$ is the clause ϕ omitting all variables not in S.
Since Φ is false and reduced the set of clauses $\Phi(x_1) = \{\phi_1(x_1), \ldots \phi_m(x_1)\}$ does not contain complementary literals, i.e. for each literal L in $\phi_i(x_1)$ the complement $\bar{L}$ does not occur in $\Phi(x_1)$. Otherwise the empty clause could not be derived by Q-unit-resolution.
Then we can remove the x_1-variables in Φ without effect to the truth of Φ obtaining the formula

$$\Phi' = \exists y_1 \forall x_2 \exists y_2, \ldots \forall x_k \exists y_k (\phi'_1 \wedge \ldots \wedge \phi'_m)$$

We define

$$\text{Part}(y_1) := \{\phi'_i | 1 \leq i \leq m \text{ and } \phi'_i(y_1) \text{ is not empty}\}$$
$$\text{Rest }(y_1) := \{\phi'_1, \ldots, \phi'_m\} - \text{Part}(y_1).$$

Since Φ' is reduced for each $\phi \in$ Part(y_1) the subclause $\phi(y_1)$ can be derived applying the Q-unit-resolution with Rest(y_1) $\cup\, \phi$.
Thus, $\text{Rest}(y_1) \cup \{\phi(x_2, y_2, \ldots, y_k)\} \vdash_{\overline{Q-U-Res}} \sqcup$ and the prefix is in Π_{k-1}. Applying the induction hypothesis we know that $\phi(y_1)$ can be derived in length $(\text{Rest}(y_1) \cup \{\phi\})^{k-1}$ Q-unit-resolution steps.

Altogether we obtain

$$\sum_{\phi \in \, Part(y_1)} \text{length } ((\text{Rest}(y_1) \cup \phi)^{k-1}$$

Q-unit-resolution steps to generate $\Phi'(y_1)$ and then length ($\Phi'(y_1)$) resolution steps to derive the empty clause. Since Part(y_1) must contain at least two clauses, otherwise Φ' is not reduced and false, we obtain our desired upper bound length $(\Phi')^k$.

q.e.d.

Theorem 3.3 *For each fixed $k \geq 1$ the evalution problem for Π_k(extended Horn) is coNP-complete.*

Proof: The evalution problem for Π_k(extended Horn) belongs to coNP, because the falsity can be decided nondeterministically by choosing at most length $(\phi)^k$ Q-unit-resolution steps (see theorem 3.2).
Since Π_1(extended Horn) is coNP-complete we have proved our desired result.

q.e.d.

This result can not be improved by means of Q-Resolution only. The following lower bound for extended quantified Horn formulas is shown in [KaKB 90]:

Theorem 3.4 *There exist quantified extended Horn formulas $\Phi_t (t \geq 1)$ of length $18t + 1$, which are false and a refutation to the empty clause requires at least 2^t Q-resolution steps.*

The following definition is needed in the proof of the next theorem: A *renaming* is the replacing of all occurences of some literals by their complements. A renaming f is a one-to-one mapping $Lit(\alpha) \to Lit(\alpha)$, such that for each $L \in Lit(\alpha) : f(L) \in \{L, \neg L\}$ and $f(L) = \neg f(\neg L)$.

Lemma 3.5 : *For each satisfiable formula α in 2-CNF a renaming f can be found in $O(length(\alpha))$ such that $f(\alpha)$ is a Horn formula.*

Proof: Since α is satisfiable and in 2-CNF a satisfying truth assignment $\mathcal{J}$ for α can be found in $O(length(\alpha))$ [APT 78].
Now the renaming is defined as

$$f(A) = \begin{cases} \bar{A} & \text{if} \quad \mathcal{J}(A) = 1 \\ A & \text{if} \quad \mathcal{J}(A) = 0 \end{cases} \qquad \text{for each variable } A \text{ of } \alpha.$$

For at least one literal L_i in each clause $\mathcal{J}(L_i) = 1$, so we can analyse the following cases with the 5 possible structures of 2-CNF clauses

$$\begin{array}{lrcl} 1) & f(B \vee C) = (\bar{B} \vee f(C)) & or & f(B \vee C) = (\bar{C} \vee f(B)) \\ 2) & f(\bar{B} \vee \bar{C}) = (\bar{B} \vee f(\bar{C})) & or & f(\bar{B} \vee \bar{C}) = (\bar{C} \vee f(\bar{B})) \\ 3) & f(B \vee \bar{C}) = (\bar{B} \vee f(\bar{C})) & or & f(B \vee \bar{C}) = (\bar{C} \vee f(B)) \\ 4) & f(B) = (\bar{B}) & & \\ 5) & f(\bar{B}) = (\bar{B}) & & \end{array}$$

In each case the result is a Horn-formula.

q.e.d.

Theorem 3.6 *For each fixed $k \geq 1$ the evalution problem for Π_k(extended 2-CNF) is coNP-complete.*

Proof: Π_1(extended 2-CNF) is known to be coNP-complete, because of theorem 3.1.

Now we will show by induction on k that the complement of the evalution problem for Π_k(extended 2-CNF) lies in NP.

Let be given $k > 1$. By induction hypothesis there is a nondeterministic algorithm NA_{k-1} deciding whether an arbitrary formula Φ_{k-1} in Π_{k-1}(extended 2-CNF) is false.
We generate for each formula Φ_k in Π_k(extended 2-CNF) nondeterministically in $O(\text{length}(\Phi_k))$ time two formulas $\Phi_{k-1,1}, \Phi_{k-1,2}$, such that $\Phi_{k-1,1}, \Phi_{k-1,2} \in \Pi_{k-1}$(extended 2-CNF), so that both formulas are false if Φ_k is false and length $(\Phi_{k-1,i}) <$ length (Φ_k) for $i = 1, 2$.

Let be given

$$\Phi_k = \forall x_1 \exists y_1 \ldots \forall x_k \exists y_k (\phi_1 \wedge \ldots \wedge \phi_m)$$

Remember that $\phi(S)$ with a set S of variables is the clause ϕ omitting all variables not in S and $\Phi(S) = \{\phi(S) | \phi \in \Phi\}$.

Step 1: [Building a reduced formula]

If Φ_k is false we nondeterministically obtain a reduced and false formula Φ'_k. If Φ_k is true the generated formula remains true. In the following whenever we observe that the resulting formula is true or not reduced we halt and nothing is known.

If a formula Φ'_k is reduced and false then each clause is necessary to derive the empty clause. Then $\Phi'_k(x_1)$ does not contain complementary literals.

Step 2: [Deletion of x_1, renaming of y_1]

If $\Phi'_k(x_1)$ contains no complementary literals, then remove all x_1-variables in Φ'_k obtaining a formula

$$\Phi''_k = \exists y_1 \forall x_2 \ldots \forall x_k \exists y_k (\phi_1 \wedge \ldots \wedge \phi_r).$$

In case of complementary literals in $\Phi'_k(x_1)$ Φ'_k is not reduced or true, so we halt.

Let be $\Phi''_k(y_1)_i$ the set of clauses in $\Phi''_k(y_1)$ with exactly i literals.

For a non-empty set $\Phi''_k(y_1)_2$ the formula Φ is false, if $\Phi''_k(y_1)_2$ is not satisfiable. In case of satisfiability there is some renaming, i.e. literals L may be replaced by $\overline{L}$ simultanously, such that the resulting formula $\Phi''_{k,H}(y_1)$ is a Horn formula. Since $\Phi''_k(y_1)_2$ contains 2-clauses only a renaming can be found in $O(\text{length}(\Phi''_k(y_1)_2))$ time (see Lemma 3.5).

Now in Φ''_k we replace $\Phi''_k(y_1)$ by the obtained Horn formula $\Phi''_{k,H}(y_1)$. The resulting formula is denoted as Φ'''_k.

Step 3: [Simplification of $\Phi'''_k(y_1)$]

If $\Phi'''_k(y_1)_2$ is empty then only 1-clauses occur in $\Phi'''_k(y_1)$ and $\Phi'''_k(y_1)_1$ must contain complementary literals, otherwise we halt, because Φ'''_k is not reduced or true.
If $\Phi'''_k(y_1)_2$ is not empty and assumed to be satisfiable we can simplify $\Phi'''_k(y_1)$ as follows:

All X-variables in the clauses of $\Phi'''_k(y_1)_2$ are not before any Y-variable.

Each clause in $\Phi'''_k(y_1)_2$ is an implication of the form $(\mathrm{y}_{i_1} \leftarrow \mathrm{y}_{i_2})$ or a negated clause of the form $(\bar{\mathrm{y}}_{i_1} \vee \bar{\mathrm{y}}_{i_2})$ for some $\mathrm{y}_{i_1}, \mathrm{y}_{i_2} \epsilon y_1$. Each clause ϕ_s in Φ'''_k with exactly one literal $\overset{(-)}{\mathrm{y}}_i \in y_1$ can be seen as a sort of unit-clause of the form $(\overset{(-)}{\mathrm{y}}_i \vee \phi_s(x_2 \ldots y_k))$. A resolution of such a clause $(\overset{(-)}{\mathrm{y}}_i \vee \phi_s(x_2 \ldots y_k))$ with a clause in $\Phi'''_k(y_1)_2$ results in a clause $(\overset{(-)}{\mathrm{y}}_j \vee \phi_s(x_2 \ldots y_k))$ with $\mathrm{y}_j \in y_1$. For all clauses $\phi_s = (\overset{(-)}{\mathrm{y}}_i \vee \phi_s(x_2 \ldots y_k)) \in \Phi'''_k$ with exactly one y-variable $\mathrm{y}_i \in y_1$ all resulting clauses $(\overset{(-)}{\mathrm{y}}_j \vee \phi_s(x_2 \ldots y_k))$ from resolution with clauses in $\Phi_k'''(y_1)_2$ can be produced in linear time by means of unit resolution. If we add these clauses to Φ_k''', we can choose nondeterministically two clauses $\phi_s = (\mathrm{y}_i \vee \phi_s(x_2 \ldots y_k))$ and $\phi_t = (\bar{\mathrm{y}}_i \vee \phi_t(x_2 \ldots y_k))$ with $\mathrm{y}_i \in y_1$. All other clauses containing variables from y_1 will be removed.

If Φ_k''' is false than there is some selection such that the resulting formula $\Phi_k'^{\mathrm{v}}$ is false and if Φ_k''' is true $\Phi_k'^{\mathrm{v}}$ remains true.

The obtained formula $\Phi_k'^{\mathrm{v}}$ has the form
$\exists \mathrm{y}_1 \forall x_2 \ldots \forall x_k \exists y_k((\mathrm{y}_1 \vee \phi_1(x_2 \ldots y_k)) \wedge (\bar{\mathrm{y}}_1 \vee \phi_2(x_2 \ldots y_k)) \wedge \phi_3 \wedge \ldots \wedge \phi_t)$,
where $\phi_3, \ldots, \phi_t$ does not contain the variable y_1.

We can divide the formula $\phi_k'^{\mathrm{v}}$ into two formulas $\Phi_{k-1,1}'$ and $\Phi_{k-1,2}'$ with free variables as follows:
$\Phi_{k-1,i}' = \forall x_2 \ldots \exists y_k : ((\mathrm{y}_1^{\epsilon_i} \vee \phi_i(x_2 \ldots y_k)) \wedge \phi_3 \wedge \ldots \wedge \phi_t)$,
where $\mathrm{y}_1^{\epsilon_1} = \mathrm{y}_1$ and $\mathrm{y}_1^{\epsilon_2} = \bar{\mathrm{y}}_1$.

If $\Phi_k'^{\mathrm{v}}$ is false and reduced then $\Phi_{k-1,1}'$ resp. $\Phi_{k-1,2}'$ must imply y_1 resp. $\bar{\mathrm{y}}_1$. Then it remains to decide whether

$$\Phi_{k-1,i} = \forall x_2 \ldots \exists y_k(\phi_i(x_2 \ldots y_k) \wedge \phi_3 \wedge \ldots \wedge \phi_t)$$

are false.

Now we can apply our nondeterministic algorithm NA_{k-1} with $\Phi_{k-1,i}(i = 1, 2)$.

Adding the above steps to NA_{k-1} we obtain a nondeterministic algorithm NA_k, which decides in polynominal time, whether a formula Φ_k in Π_k(extended 2-CNF) is false.

q.e.d.

Instead of restricting the existential part of the formulas to Horn-clauses resp. 2-clauses, one may also consider restrictions of the universal variables of the clauses. The following theorem shows, that from such a restriction, no gain in complexity will result.

Theorem 3.7 *The evalution problem for quantified Boolean formulas, for which each clause contains at most one universal variable is PSPACE-complete.*

Proof: Let be given any quantified Boolean formula $\Phi = \forall x_1 \exists y_1 \ldots \forall x_k \exists y_k(\phi_1 \wedge \ldots \wedge \phi_m)$, where $x_i = \mathrm{x}_{n_{i-1}+1} \ldots \mathrm{x}_{n_i}$, $1 \leq i \leq k$ and $n_0 = 1$.

Now we introduce new existential variables $\mathrm{z}_1, \ldots, \mathrm{z}_{n_k}$ and define

$$\begin{aligned}\Phi^* = {} & \forall \mathrm{x}_1 \exists \mathrm{z}_1 \forall \mathrm{x}_2 \exists \mathrm{z}_2 \ldots \forall \mathrm{x}_{n_1} \exists \mathrm{z}_{n_1} \exists y_1 \ldots\ldots \\ & \forall \mathrm{x}_{n_{k-1}+1} \exists \mathrm{z}_{n_{k-1}+1} \ldots \forall \mathrm{x}_{n_k} \exists z_{n_k} \exists y_k : \left(\bigwedge_{1 \leq i \leq n_k} ((\mathrm{x}_i \vee \bar{\mathrm{z}}_i) \wedge (\bar{\mathrm{x}}_i \vee \mathrm{z}_i)) \right) \\ & \wedge \bigwedge_{1 \leq j \leq n} \phi_j[\mathrm{x}_1/\mathrm{z}_1, \ldots, \mathrm{x}_{n_k}/\mathrm{z}_{n_k}]),\end{aligned}$$

where $\phi_j[\mathrm{x}_1/\mathrm{z}_1, \ldots, \mathrm{x}_{n_k}/\mathrm{z}_{n_k}]$ is the clause obtained by replacing in ϕ_j each occurence of x_i resp. $\bar{\mathrm{x}}_i$ by the variable z_i resp. $\bar{\mathrm{z}}_i$.

As easily can be seen Φ is true iff Φ^* is true.

Φ^* contains clauses with at most one universal variable only. Thus we have proved our desired result, because the evalution problem for quantified Boolean formulas is PSPACE-complete and our transformation can be performed in polynominal time.

q.e.d.

4 Conclusion

We have shown that the evalution problem for the subclasses Π_k(extended Horn) and Π_k(extended 2-CNF) of quantified Boolean formulas for any fixed $k \geq 1$ is coNP-complete. We hope these results will help to solve the complexity of the evalution problem for extended quantified Horn resp. 2-CNF formulas, that still remains open.

References

[APT 78] B. Aspvall, M. F. Plass and R.E. Tarjan: *A Linear-Time Algorithm for Testing the Truth of Certain Quantified Boolean Formulas*, Information Processing Letters, Volume 8, number 3, 1978

[DoGa 84] W. F. Dowling and J. H. Gallier: *Linear-Time Algorithms For Testing The Satisfiability Of Propositional Horn Formulae*, J. of Logic Programming, 1984

[GJ 79] M. R. Garey and D. S: Johnson: *Computers and Intractability, A Guide to the Theory of NP-Completeness*, W. H. Freemann and Co., San Fransicao, 1979

[Ha 85] A. Haken: *The intractability of resolution*, Theoret. Comput. Sci. 39, 1985, 297–308

[KaKB 90] M.Karpinski, H.Kleine Büning: *Resolution for Quantified Boolean Formulas,* submitted for publication

[KKBS 87] M. Karpinski, H. Kleine Büning and P.H. Schmitt: *On the computational complexity of quantified Horn clauses*, Lecture Notes in Computer Science 329, Springer-Verlag, 1987

[StM 73] L. J. Stockmeyer and A. R. Meyer: *Word problems requiring exponential time*, Proc. 5th Ann. ACM Symp. Theory Computer, 1973