PARALLEL COMPLEXITY FOR MATCHING RESTRICTED TO DEGREE DEFINED GRAPH CLASSES

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1. Introduction

The parallel complexity of the perfect matching problem has become one of the most intriguing computation problems after [KUW] and [MVV] that it is in RNC (randomized NC). For a general overview of NC-classes see [Co]. The problem designing deterministic NC-algorithms for the perfect matching problem remains wide open. For special graph classes, the NC -algorithms have been developed [DK1, GK , HM, KVV, LPV]. For other graph classes we know that their matching problem is as hard as the general case [DK1, DK2, KL].

For example [LPV] presented an NC^2-algorithm for the construction of a perfect matching in bipartite regular graphs. It is a known theorem (see for example [Ai]) that each bipartite regular graph has a perfect matching. More or less we want to base on the following hypothesis:

1) Degree defined classes, s.t. each graph of them has a perfect matching, have an NC^2-construction algorithm.
2) For degree defined classes, s.t. not every graph of them has a perfect matching, the perfect matching problem is as hard as the general matching problem.

We know for example that each graph of minimal degree 1/2 |G| and an even number of vertices has a perfect matching (see for example [Bo]). We call such graphs *high degree graphs* or *dense*. We know also that each graph which does not contain a bridge (an edge whose deletion enlarges the number of connected components) and which is 3-regular has a perfect matching (Theorem of Peterson, see for example [Ai]).

We shall show that we can construct a perfect matching for dense graphs in NC^2. For all interesting classes properly between dense graphs and bridgeless 3-regular graphs the perfect matching problem is matching hard by AC^0-reductions. Related results concerning the k-coloring problem can be found in [Ed]. That was one motivation of our paper. The construction problem for a perfect matching in 3-regular graphs remains open.

Section 2 will give some foundations and notations. In section 3 we will present a construction algorithm of a perfect matching for dense graphs. We will also consider graphs of minimal degree $\alpha|G|$, s.t. $\alpha<\frac{1}{2}$.

Section 3 will present completeness results for low degree graphs. That includes 2-connected graphs of maximal degree 3, 2-connected 4-regular graphs, and 3-regular graphs.

2. Notations and Foundations

2.1. *General Terminology on Graphs:*
A graph G=(V,E) consists of a set V of *vertices* and a set E of *edges*. |G|=|V| is the number of vertices of G and |E| is the number of edges of G. Generally we denote the power of a set S by |S|.

2.2. A *matching* of a graph G is a subset M of E, s.t. no two edges of M have a common vertex. A matching M is called *perfect*, if each vertex of G is contained in some edge of M.

2.3. The *degree* of a vertex v is the number of edges containing v and is denoted by d(v). A graph is k-*regular* or regular of the degree k if all vertices have the same degree k.
If G is k-regular for some k then we call G regular.

2.4. An *edge coloring* of G with k colors is a map c: E -> {1...k}, s.t. $c(e_1) \neq c(e_2)$ if e_1 and e_2 have a common vertex.

THEOREM 1 (see for example [Ai] p. 135): Each regular bipartite graph of degree k has an edge coloring with k colors

2.4. Let NC^k be the class of all functions and predicates, which are computable or decidable by a uniform sequence (logspace) of circuits of polynomial size and $(\log n)^k$ depth. AC^0 is the class of all functions computable by a uniform sequence of unbounded fan in circuits of polynomial size and constant depth (see [Co]). For the notion of completeness we shall use an extension of Cook's notion. We call a predicate A $AC^0(NC^k)$-*hard* for B if there is an AC^0 (NC^k)-mapping which reduces B to A, that means there is an AC^0 (NC^k)-mapping f, s.t. $B=f^{-1}[A]$. We say that A is AC^0-*complete* for B if additionally there is a mapping g vice versa reducing A to B. This notions are similar to the usual notion of NP-completeness of graphisomorphism completeness. All local replacement reductions are in AC^0 (a usual method of NP-completeness proofs (see [GJ]).

2.5. A first known result concerning the complexity of matching in parallel is the following.

THEOREM 2 [LPV]: An edge coloring of k colors of a k-regular bipartite graph can be constructed in NC^2.

Now each color forms a perfect matching.

Corollary 3: Each regular bipartite graph has a matching which can be constructed in NC^2.

One problem is, how to construct a perfect matching for regular but not necessarily bipartite graphs. We know only the following.

THEOREM 4 (Peterson, see [Ai]): Each 2-connected (and therefore bridgeless) 3-regular graph has a perfect matching.

We call an edge a *bridge* if its deletion enlarges the number of connected components.

2.6. For graphs of high degrees (not necessarily regular) we know the following.

THEOREM 5 (see for example [Bo]): Each graph G of minimal degree $|G|/2$ has a matching of power $[|G|/2]$. $[m]$ is here the greatest integer not exceeding m.

That means:

Corollary 6: Each graph G of an even number of vertices and a minimal degree $|G|/2$ has a perfect matching.

Call graphs G with a minimal degree $|G|/2$ *dense* or *high degree graphs*.

2.7. In the whole paper we will only consider graphs of an even number of vertices.

3. Construction of a Perfect Matching for Dense Graphs

In this section we will present an NC^2-algorithm constructing for each dense graph a perfect matching. The key of the algorithm is the following result.

Lemma 6 [Lu]: There is an NC^2-algorithm constructing for each graph a nonextendible (also called maximal) independent set.
Remark [Lu]: The above algorithm implemented on an EREW P-RAM (parallel random access machine without concurrent read and concurrent write) needs $O(|V|^2\ |E|)$ processors in the worst case.

An immediate consequence is the following.

Lemma 7([Lu] see also [KW]): For each graph a nonextendible matching can be constructed in NC^2.

Remark: For the construction of a nonextendible matching we need $O(|E|^4)$ processors in the worst case.

Lemma 7 can be derived from Lemma 6 by constructing from a graph G=(V,E) a graph G'=(V',E'), s.t. V'=E and two edges of E are joint by an edge in G' iff they have a common vertex. Clearly a nonextendible matching in G is the same as a nonextendible independent set in G'. The number of processors which are needed for the construction of a nonextendible matching can be easily derived from the construction of G'.

Now we can state the main result of this section.

THEOREM 8: For each dense graph of an even number of vertices a perfect matching can be constructed in NC^2.

For the *proof* we state a straight line algorithm, s.t. each single step can be executed in NC^2:

Input: a dense graph G=(V,E).
First step: Compute any nonextendible matching M_1 of G.
Comments:
Each edge contains at least one vertex appearing in M_1, otherwise M_1 would be extendible.
At least $\frac{|G|}{2}$ *vertices belong to an edge of* M_1. We may assume that there is a vertex v not belonging to an edge of M_1. But v is joined by an edge to at least $|G|/2$ vertices. They belong all to an edge of M_1.
End of comments

Second step: Let $\{x_1,\ldots,x_{2k}\}$ be the set of vertices of G not belonging to an edge of M_1 and define $G'=(V',E')$ as follows:
The vertex set V' consists of the edges of M_1 and of the unordered pairs $\{x_{2i-1},x_{2i}\}, i=1,..,k$. The edge set is defined as follows:
$\{x_{2i-1},x_{2i}\}$ and $\{y,z\}\in M_1$ are joined by an edge in E' iff
$\{x_{2i-1},y\}$ and $\{x_{2i},z\} \in E$ or
$\{x_{2i-1},z\}$ and $\{x_{2i},y\} \in E$.

Comment: Note that G' is bipartite. End of comment.

Third step: Compute a nonextendible matching M_2 of G'.

Comments:
Each vertex of G' *of the form* $\{x_{2i-1},x_{2i}\}$ *belongs to an edge of* M_2.
We shall prove this claim by the following statement.

Lemma 9: Let k be defined as above as the number of pairs $\{x_{2i-1},x_{2i}\}$. The degree of each G'-vertex x_{2i-1},x_{2i} is at least k.

Proof of the lemma:
Set $M_1=\{\{u_i,v_i\}:i=1,..,p\}$.
For each G-vertex x not belonging to an edge of M_1 set
$t_x:=\{i:$exactely one of u_i and v_i is joined by an edge with x$\}$ and
$h_x:=\{i:x$ is joined by an edge with u_i and $v_i\}$.

By the fact that G is dense we get the following inequality:

$|t_x|+2|h_x|\geq k+p$ (note that G has 2k+2p vertices).

For simplicity set $x:=x_{2i-1}$ and $y:=x_{2i}$. Then at least the $\{u_i,v_i\}$, s.t. $i\in J:=(t_x\cap h_y)\cup(h_x\cap h_y)\cup(t_y\cap h_x)$ are joined by an edge of G' with $\{x,y\}$. Note that the three components of the union are pairwise disjoint.
Set $j:=|h_x\cap h_y|$. We will state the following equalities and inequalities which are easily checked:
i) $t_x\cap h_y=t_x\cap(h_x\cup h_y)$ (because $h_x\cap t_x=\emptyset$)
ii) $|t_x\cup h_x\cup h_y|\leq p$
iii) $|t_x\cup h_x\cup h_y|=|t_x|+|h_x\cup h_y|-|t_x\cap(h_x\ h_y)|$

Therefore:

$|t_x|+|h_x \cup h_y|-p \leq |t_x \cap (h_x \cup h_y)| = |t_x \cap h_y|$

But that means

$|t_x \cap h_y| \geq |t_x|+|h_x|+|h_y|-j-p$ and analogeously

$|t_y \cap h_x| \geq |t_y|+|h_y|+|h_x|-j-p$

Therefore

$|J| = |t_x \cap h_y|+|t_y \cap h_x|+j \geq 2(k+p)-2j-2p+j = 2k-j.$

If j is greater than k then J has trivially a power greater than k. In the other case by the last inequality $|J| \geq k$. But that means that at least k pairs $\{u_i, v_i\}$ are joined by an edge with $\{x,y\}$.
End of comments.

Fourth step: For each $\{x_{2i-1}, x_{2i}\}$as above consider the $\{u,v\}$ which is joined by an edge of M_2. W.l.o.g $\{x_{2i-1}, u\}$, $\{x_{2i}, v\} \in E$. Delete $\{u,v\}$ from M_1 and add $\{x_{2i-1}, u\}$ and $\{x_{2i}, v\}$ to M_1.

Comment: M_1 is changed to a perfect matching. End of comment.

Last step: Output M_1.

The correctness of the algorithm follows from the comments. The algorithm defines an NC^2-function because each step is in NC^2.

QED

Remark: If we want to check the number of processors we have to check the number of processors of each step and to take the maximum. We need a large number of processors in the first and in the third step. But G' has only $|G|/2$ vertices and at most $|E|/2$ edges. Therefore the number of processors can be bounded by $O(|E|^4)$ for the whole algorithm.

The next question is the parallel complexity of matching for graphs of a minimal degree $\alpha|G|$, s.t. $\alpha \leq \frac{1}{2}$.

THEOREM 10: For $\alpha < \frac{1}{2}$ the existence problem for a perfect matching restricted to graphs G=(V,E), s.t. minimal degree is $\alpha|V|$, is AC^0-hard for the general matching problem.

Proof:

Let G=(V,E) be any graph. We construct a graph G'=(XÜYÜV,E') as follows: X forms a complete subgraph of G' and Y forms an independent set in G'. Each vertex of X and each vertex of V are joined by an edge in E' and each vertex of X and each vertex of Y are joined by an edge E'. Vertices of V are joined by an edge in G' iff they are joined by an edge in G. X and Y have the same power.

Claim: G has a perfect matching iff G' has a perfect matching:

Let M' be a perfect matching of G'. Then M' defines a bijection between X and Y, because Y is independent and all edges of Y go to X. Therefore no edges between V and X are in M'. That means M' restricted to V defines a perfect matching on G.

Let M be a perfect matching on G and f be a bijection between X and Y. Then clearly a perfect matching on G' is defined.

The minimal degree of G' is the power of X and by definition $|G'|=2|X|+|G|$. Set X as large that

$$\frac{|X|}{2|X|+|G|}=\alpha$$

But that means

$$|X|=\frac{\alpha}{(1-2\alpha)}|G|.$$

But for $\alpha<\frac{1}{2}$ we have $\frac{\alpha}{1-2\alpha}>0$.

QED

4. Matching on Low Degree Graphs

In this section we will give some results on the parallel complexity of graphs of maximal degree 3 and 4. We know by the theorem of Peterson that each bridgeless 3-regular graph has a perfect matching. We shall prove that the perfect matching problem on all interesting upper classes is AC^0-hard for the general perfect matching problem.

THEOREM 11: The perfect matching problem restricted to 2-connected graphs of maximal degree 3 is AC^0-hard for the general perfect matching problem.

Proof:

Given any 2-connected graph G=(V,E). We construct a maximal degree 3 graph G'=V',E') as follows:

For each vertex v of G let $e_1^v,\ldots,e_k^v$ be an enumeration of its adjacent edges. Replace v by vertices $u_1^v,\ldots,u_k^v$ and $w_1^v,\ldots,w_{k-1}^v$ of V'. The edges of G' are defined as follows:

For each i<k: $\{u_i^v,w_i^v\}$, $\{w_i^v,u_{i+1}^v\}\in E'$, and if $e_i^v=e_j^{v'}$ is an edge of G then $\{u_i^v,u_j^{v'}\}\in E'$.

Clearly the construction of G' from G can be done in AC^0 and G' is 2-connected. We have to prove that G has a perfect matching if and only if G' has a perfect matching. Let M be a perfect matching of G. If $e=e_i^{v''}=e_j^{v'}$, replace e by $\{u_i^{v''},u_j^{v'}\}$ and for k=i and j and v=v" and v' resp. set $\{u_m^v,w_m^v\}\in M'$ for m<k and $\{w_{m-1}^v,u_m\}\in M'$ for m>k. This defines a perfect matching on G'.

On the other hand let M' be a perfect matching on G'. Since $V^v:=\{u_i^v:i=1,\ldots,k\}\cup\{w_i^v:i=1,\ldots,k-1\}$ is odd, at least one V^v leaving edge $e=\{u_j^v,e_k^{v'}\}$ is in M'. But then $\{u_{j-1},w_{j-1}\}\in M'$ (that is the only remaining edge of M' containing w_{j-1}) and so on $\{u_i,w_i\}\in M'$ for all i<j. Analogeously $\{w_{i-1},u_i\}\in M'$ for all i>j. That means that exactely one V^v leaving edge, which is represented by an edge e of G leaving v, is in M'. Set $e\in M$. Then M is a perfect matching on G.

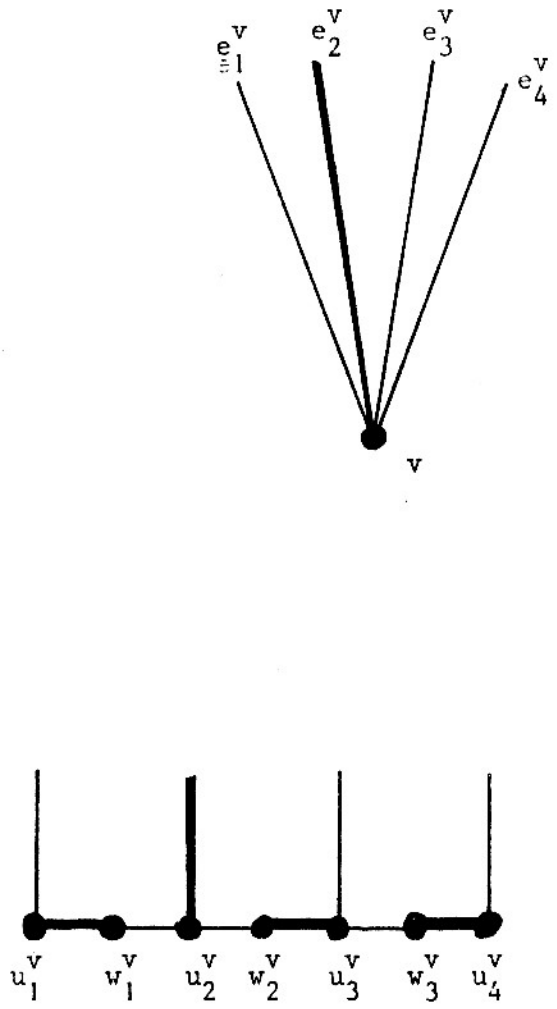

Figure 1: Reduction of the perfect matching problem to the perfect matching problem for graphs of maximal degree 3 (bold edges are in a perfect matching)

Now we want to consider regular graph structures of small degree.

Theorem 12: Perfect matching for 2-connected 4-regular graphs is AC^0-hard for the general perfect matching problem.

Proof: We construct an AC^0-reduction from the matching problem restricted to 2-connected graphs of maximal degree 3. We give at first a reduction to graphs of degree 3 or 4.
Consider any 2-connected graph $G=(V,E)$ of maximal degree 3.
Let $H_5(u_1,u_2)$ be the 5-clique without the edge $\{u_1,u_2\}$.
Let v be a vertex of degree 2 with the neighbors v_1 and v_2.
Replace v by $H_5(u_1,u_2)$ and join the pairs $\{v_1,u_1\}$ and $\{v_2,u_2\}$ by an edge. Call the graph constructed in that way $G'=(V',E')$.
G' has only degrees 3 and 4.
Claim: G' has a perfect matching if and only if G has a perfect matching:
Let M' be a perfect matching of G'. Then exactely one edge leaving $H_5(u_1,u_2)$ is in M' and a perfect matching on G is defined. Vice versa one has only to enlarge the matching M of G by matchings on copies of $H_5(u_1,u_2)-\{u_i\}$ or $H_5(u_1,u_2)-\{u_2\}$, which are both 4-cliques.
The next step is to reduce the perfect matching problem for graphs of degree 3 or 4 which are 2-connected to the perfect matching problem for 4-regular 2-connected graphs. W.l.o.g. we have to consider only graphs G' of an even number of vertices. But then we have an even number of vertices of degree 3. Let $(u_i:i=1,\ldots,2p)$ be an enumeration of the vertices of degree 3. For $j=1,\ldots,p$ let $H_j(x_1,x_2)$ and $H_j(y_1,y_2)$ be copies of $H_5(u_1,u_2)$ and s_j,t_j two additional vertices.
Join the pairs $\{u_{2j-1},s_j\}$, $\{u_{2j},t_j\}$, $\{s_j,t_j\}$, $\{s_j,x_1\}$, $\{s_j,y_1\}$, $\{t_j,x_2\}$, $\{t_j,y_2\}$ by an edge. Call the graph constructed in that way, together with the edges of G', G". Clearly G" is 2-connected and 4-regular.
Claim: G" has a perfect matching iff G' has a matching:
For each j the number of vertices of $V_j:=H_j(x_1,x_2)\cup H_j(y_1,y_2)\cup\{s_j,t_j\}$ has an even number of vertices and two leaving edges on s_j and t_j.
But that means: both leaving edges are in a perfect matching or none of both is in the perfect matching. But in the first case the $H_j(x_1,x_2)$ and $H_j(y_1,y_2)$ are isolated which is a contradiction.
Therefore each perfect matching on G" defines a perfect matching on G'.

is transformed into:

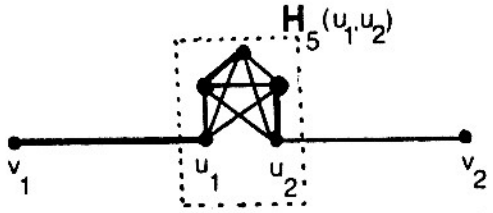

Figure 2: Reduction of the perfect matching problem for graphs of maximal degree 3 to graphs of degree 3 or 4.

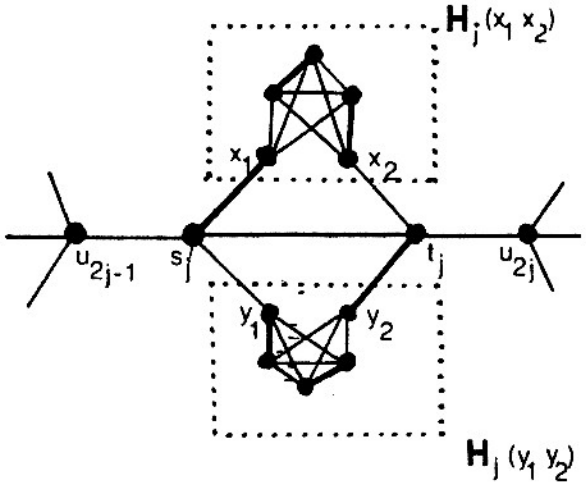

Figure 3: Reduction of the perfect matching problem for graphs of degree 3 or 4 to 4-regular graphs by enlarging the degree of two vertices of degree 3 at once (bold edges belong to a perfect matching)

On the other way round enlarge any matching M' on G' by edges $\{s_j,x_1\},\{t_j,y_2\}$ and natural enlargements on $H_j(x_1,x_2)-\{x_1\}$ and $H_j(x_1,x_2)-\{x_2\}$. Also this construction can be done in AC^0.

QED.

The other immediate upper class of 3-regular 2-connected graphs are simply the 3-regular graphs (which are not necessarily 2-connected).

THEOREM 13: The perfect matching problem for 3-regular graphs is AC^0-hard for the general perfect matching problem.

Proof: Consider any graph of maximal degree 3 which is 2-connected. We know that the number of vertices of odd degree is even. But that are exactely the vertices of degree 3. But then also the number of vertices of even degree is even. That are exactely the vertices of degree 2. Let $(u_i:i=1...2k)$ an enumeration of the vertices of degree 2 in a maximal degree 3 graph G'. Construct a 3-regular graph G" as follows:
G" contains the vertices and edges of G' and for each j=1,...,k additional verices $q_1^j,....,q_6^j$ and additional edges $\{u_{2j-1},q_1^j\},\{u_{2j},q_1^j\},\{q_1^j,q_2^j\},\{q_2^j,q_3^j\},\{q_3^j,q_4^j\},\{q_4^j,q_5^j\},\{q_5^j,q_6^j\},\{q_6^j,q_2^j\}$, $\{q_3,q_5\},\{q_4,q_6\}$. The set $\{q_k^j:j\geq 2\}$ is odd and has as only leaving edge $\{q_1^j,q_2^j\}$, which is forced to be in any perfect matching of G". Therefore no $\{q_k^j : j\geq 1\}$ -leaving edge is in a perfect matching, and a perfect matching on G' is defined.
On the other way round let M' be a perfect matching on G'. Add the edges $\{q_1,q_2\},\{q_3,q_4\},\{q_5,q_6\}$. This defines a perfect matching on G".

QED.

5. Final Remarks

The construction of a perfect matching for bridgeless 3-regular graphs in parallel remains an open problem. We claim that this works in NC^2. But we could not prove it. We refer to [KUW2].

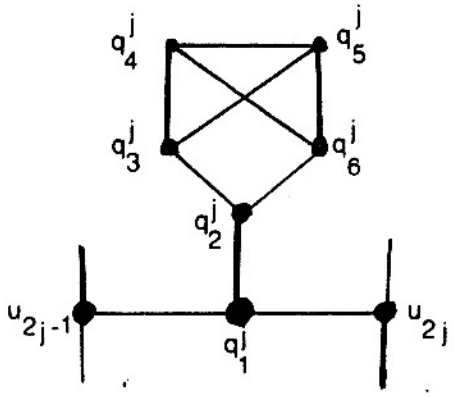

Figure 4: Reduction of the perfect matching problem restricted graphs of maximal degree 3 to the perfect matching problem for 3-regular graphs by enlarging the degree of vertices of degree 2, say u_{2j-1}, u_{2j} (bold edges belong to a perfect matching)

This paper deals with the question of equivalence of existence and construction problems.

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