Perfect Matching for Regular Graphs is AC^0-Hard for the General Matching Problem

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Abstract.

We shall prove that the perfect matching for regular graphs (even if restricted to degree 3 and 2-connected 4-regular graphs) is AC^0-equivalent with the general perfect matching problem for arbitrary graphs.

1 Introduction

The parallel complexity of deciding the existence of a perfect matching in a graph is an open problem. A *perfect matching* M of a graph $G = (V, E)$ is a

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set of edges from E which cover all vertices, so that no two edges of M have a common vertex. Randomized NC-algorithms are known [KUW 1], [MVV]. Also for special graph subclasses we know NC^2-algorithms for constructing a perfect matching. Examples are strongly chordal graphs [DK], bipartite graphs with a bounded permanent [GK], complements of transitive orientable graphs [HM], and *bipartite regular graphs* [LPV]. The last class motivated the question of the *parallel complexity of the perfect matching problem for regular (not necessarily bipartite) graphs*. The arguments of [LPV] fail for non-bipartite regular graphs. We know that each regular bipartite graph has a perfect matching and that we can color the edges with as many colors as the degree of every vertex [LPV]. This is not true for non-bipartite regular graphs. There is only one known additional result of Peterson (cf., e.g., [Ai] or [Bo 1]).

Proposition. (Peterson's Theorem) Every 2-connected 3-regular graph has a perfect matching.

The above result is existential and does not give a method for constructing a matching.

In this paper we prove that, whenever we relax one of the two conditions (being "*2-connected*" or "*3-regular*") in Peterson's Theorem, the problems of deciding the existence and the construction of a perfect matching in resulting classes of graphs are both AC^0-equivalent to the respective perfect matching problems in general graphs. It is an open question in parallel computation whether the decision and the construction problems in perfect matching are mutually NC-equivalent (cf. [KUW 2]). In the sequel we shall use the expression "perfect matching problem" to apply (separately) to both the decision and the construction problem.

We shall prove the following surprising result:

Main Theorem. The perfect matching problem restricted to 3-regular graphs and 4-regular 2-connected graphs is AC^0-equivalent with the general perfect matching problem. The reduction uses $O(n^2)$ boolean processors.

In particular: There exists a uniform sequence of unbounded fan in circuits of polynomial size and constant depth which constructs for every graph G

a 3-regular graph G' or a 4-regular 2-connected graph G', respectively, such that G has a perfect matching if and only if G' has a perfect matching. Moreover, from a perfect matching in G' we can compute a perfect matching in G by an AC^0-algorithm.

Being AC^0 means computable by uniform unbounded fan in circuits of polynomial size and constant depth (see [Co], [KR]). An overview of parallel complexity classes can be found in [Co].

Therefore an NC-algorithm for the perfect matching problem restricted to these graph classes would induce an NC-algorithm for the general perfect matching problem. For the proof of the main theorem we need two auxiliary hardness results:

1) The AC^0-*hardness* of the perfect matching problem restricted to 2-connected graphs for the general matching problem;

2) The AC^0-*hardness* of the perfect matching problem restricted to graphs of maximal degree 3.

Section 2 will present some basic definitions which will be used for the whole paper.

Section 3 will present the auxiliary hardness results mentioned above.

In Section 4 we shall prove the main theorem.

2 Basic Definitions and Results.

A graph $G = (V, E)$ consists of a set V of *vertices* and a set E of *edges*. $|G| = |V|$ is the number of vertices of G and $|E|$ is the number of edges of G. Generally, we also denote the cardinality of a set S by $|S|$.

G is bipartite, if there is a pair (U_1, U_2) of complementary subsets of V, such that each edge of G has one end in U_1 and one end in U_2.

A *matching* of a graph G is a subset M of E, so that no two edges of M have a common vertex. A matching M is called *perfect* if each vertex of G is contained in some edge of M.

The *degree* of a vertex v is the number of edges containing v and is denoted by $d(v)$. A graph is $k-regular$ or regular of the degree k, if all the vertices have the same degree k. If G is k-regular for some k, then we call G regular.

An *edge coloring* of G with k colors is a map $c : E \to \{1 \cdots k\}$, so that $c(e_1) \neq c(e_2)$ if e_1 and e_2 have a common vertex. The class of all edges of the same color forms a perfect matching of G. The following is true for regular bipartite graphs:

Theorem 1. (cf., e.g., [Ai], p.135): Each regular bipartite graph of degree k has an edge coloring with k colors.

Let NC^k be the class of all functions and predicates which are computable or decidable by a uniform (logspace) sequence (cf. [Co]) of circuits of polynomial size and $O(\log^k n)$ depth. AC^0 is the class of all functions computable by a uniform sequence of unbounded fan in circuits of polynomial size and constant depth. We call a predicate B, AC^0-*reducible* to A, $B \leq_{AC^0} A$, if there is an AC^0-computable function f, such that $B = f^{-1}(A)$. We say that A and B are AC^0-*equivalent*, $A \equiv_{AC^0} B$, iff $B \leq_{AC^0} A$ and $A \leq_{AC^0} B$. (We also say, that A is AC^0-*hard* for B if $B \leq_{AC^0} A$; and that A is AC^0-complete for B iff $A \equiv_{AC^0} B$. These notions are similar to the notion of *graph isomorphism completeness* (cf. [BC])).

There is a well known open problem of self-reducibility of general search and decision problems in parallel computation (cf. [KUW 2]).

It is not known whether the problems of deciding the existence of a perfect matching and constructing a perfect matching are NC-equivalent. A *construction* (search) problem is defined as follows. For a given predicate P and x, construct y such that $P(x, y)$ or output 'no' if no such y exists. As an example, a perfect matching problem is a predicate $M(x, y) \Longleftrightarrow y$ is a perfect matching in a graph x.

We say that the construction problem $A(x, y)$ is $AC^0(NC^k)$-reducible to

$B(x', y')$ if there exist $AC^0(NC^k)$-computable mappings $f(x)$ and $g(x, y)$, such that for all x

(1) $\exists y \quad A(x, y) \iff \exists y' \quad [B(f(x), y')]$ (the existence problem related to A is reduced to the existence problem related to B by mapping f)

(2) $B(f(x), y') \Longrightarrow A(x, g(x, y'))$ (if the construction problem for B is in $AC^0(NC^k)$).

The notions of equivalence, hardness and completeness for the construction problems are defined analogously as for decision problems. A first known result concerning the complexity of matching in parallel was the following

Theorem 2. [LPV]: An edge coloring of k colors of a k-regular bipartite graph can be constructed in NC^2.

Since every color forms a perfect matching, we have

Corollary 1. Each regular bipartite graph has a perfect matching which can be constructed in NC^2.

One problem is, however, how to construct a perfect matching for regular but not necessarily bipartite graphs. We know only the following existential result on 2-connected regular graphs.

Theorem 3. (Peterson, see [Ai]): Each 2-connected (and therefore bridgeless) 3-regular graph has a perfect matching.

We call an edge a *bridge* if its deletion enlarges the number of connected components.

In the whole paper we shall consider only graphs with an even number of vertices.

3 Auxiliary Hardness Results.

At first we prove the following

Lemma 1. The existence and the construction problem for a perfect matching restricted to 2-connected graphs is AC^0-equivalent with the existence and the construction problem for a general perfect matching problem respectively.

PROOF. We shall use the method of "superfluous" edge systems. That means we add subgraphs H with two leaving edges e_1, e_2, both adjacent to exactly one vertex v_1 or v_2, respectively, of the old graph, so that the edges e_1 and e_2 are not in any perfect matching. We say also that v_1 and v_2 are joined by H. Consider for example the graph H (see Figure 1).

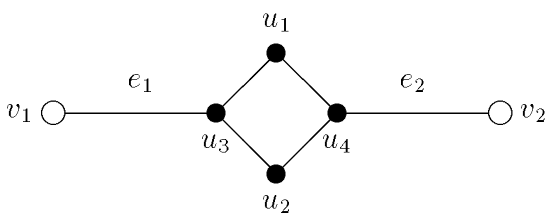

Figure 1: Graph H

Now all vertices $v_1 \neq v_2$ of G are joined by H. Then the resulting graph G' is 2-connected. Also inside H there are only two possibilities of perfect matching:$\{[u_1, u_3], [u_2, u_4]$ and $[u_1, u_4], [u_2, u_3]\}$. Therefore G has a perfect matching if and only if G' has a perfect matching. It is easily seen that we can construct a perfect matching in G from a perfect matching in G' in AC^0.

$\square$

The next result we prove is the following

Theorem 4. The existence and the construction problem for a perfect matching restricted to 2-connected graphs of maximal degree 3 is AC^0-equivalent with the existence and construction problem for a general perfect matching problem respectively.

PROOF. Given any 2-connected graph $G = (V, E)$. We construct a maximal degree 3 graph $G' = (V', E')$ as follows:

For each vertex v of G let $e_1^v, \cdots, e_k^v$ be an enumeration of its adjacent edges. Replace v by vertices $u_1^v, \cdots, u_k^v$ and $w_1^v, \cdots, w_{k-1}^v$ of V'. The edges of G' are defined as follows:

For each $i < k$: $\{u_i^v, w_i^v\}, \{w_i^v, u_{i+1}^v\} \in E'$, and if $e_i^v = e_j^{v'}$ is an edge of G then $\{u_i^v, u_j^{v'}\} \in E'$.

Clearly the construction of G' from G can be done in AC^0 and G' is 2-connected. We have to prove that G has a perfect matching if and only if G' has a perfect matching. Let M be a perfect matching of G. If $e = e_i^{v''} = e_j^{v'}$, replace e by $\{u_i^{v''}, u_j^{v'}\}$ and for $l = i$ & $v = v''$ and for $l = j$ & $v = v'$, respectively, set $\{u_m^v, w_m^v\} \in M'$ for $m < l$ and $\{w_{m-1}^v, u_m^v\} \in M'$ for $m > l$. This defines a perfect matching on G'.

On the other hand let M' be a perfect matching on G'. Since $V^v := \{u_i^v : i = 1, \cdots, k\} \cup \{w_i^v : i = 1, \cdots, k-1\}$ is odd, at least one V^v leaving edge $e = \{u_j^v, e_k^{v'}\}$ is in M'. But then $\{u_{j-1}, w_{j-1}\} \in M'$ (that is the only remaining edge of M' containing w_{j-1}) and so on $\{u_i, w_i\} \in M'$ for all $i < j$. Analogously $\{w_{i-1}, u_i\} \in M'$ for all $i > j$. That means that exactly one V^v leaving edge, which is represented by an edge e of G leaving v, is in M'. Set $e \in M$. Then M is a perfect matching on G.
The reduction of the perfect matching problem to the perfect matching problem for graphs of maximal degree 3 (bold edges are in a perfect matching) is shown in Figure 2.

It is easily seen that we can construct a perfect matching in G from a perfect matching in G' in AC^0.

□

4 Proof of the Main Theorem.

We shall prove the following result:

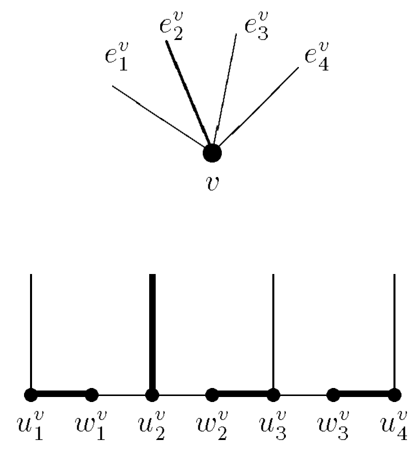

Figure 2: Reduction of the perfect matching problem

Lemma 2. The existence and the construction problem for a perfect matching restricted to 2-connected 4-regular graphs is AC^0-equivalent with the existence and construction problem for a general matching problem respectively.

PROOF. We construct an AC^0-reduction from the matching problem restricted to 2-connected graphs of maximal degree 3. At first we give a reduction to graphs of degree 3 or 4. Consider any 2-connected graph $G = (V, E)$ of maximal degree 3. Let $H_5(u_1, u_2)$ be the 5-clique without the edge $\{u_1, u_2\}$. Let v be a vertex of degree 2 with the neighbors v_1 and v_2. Replace v by $H_5(u_1, u_2)$ and join the pairs $\{v_1, u_1\}$ and $\{v_2, u_2\}$ by an edge. Call the graph constructed in that way $G' = (V', E')$. G' has only degrees 3 and 4.

Claim. G' has a perfect matching if and only if G has a perfect matching.

Let M' be a perfect matching of G'. Then exactly one edge leaving $H_5(u_1, u_2)$ is in M' and a perfect matching on G is defined. Vice versa one has only to enlarge the matching M of G by matchings on copies of $H_5(u_1, u_2) - \{u_1\}$ or $H_5(u_1, u_2) - \{u_2\}$, which are both 4-cliques.

Remark. A perfect matching in G can be constructed from a perfect matching in G' in AC^0 in a straightforward way.

The next step is to reduce the perfect matching problem for graphs G' of degree 3 or 4 which are 2-connected to the perfect matching problem for 4-regular 2-connected graphs. W.l.o.g. we have to consider only graphs G' of an even number of vertices. But then we have an even number of vertices of degree 3. Therefore there are either no vertices of degree 3 or at least two. For the case that there are no vertices of degree 3 let $G'' := G'$. For the case that there are vertices of degree 3 let G'_1, G'_2 be two copies of G'. Let u be a vertex of degree 3 in G'. Let u_1 and u_2 be the corresponding vertex in G'_1 and G'_2, respectively. Join u_1, u_2 by the graph H shown in Figure 3.

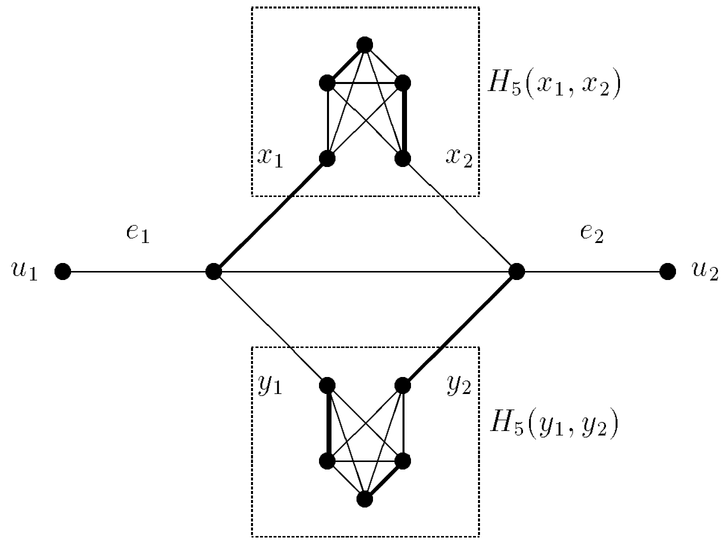

Figure 3: Graph H

The resulting graph G'' is 4-regular. It is also 2-connected because we have at least two pairs of vertices (u_1, u_2) in $G'_1 \cup G'_2$ which have to be connected by H. The subgraphs $H_5(x_1, x_2)$, $H_5(y_1, y_2)$ have the same behavior as vertices connecting to perfect matching. As in the proof of the hardness of the perfect matching problem restricted to 2-connected graphs, it is easily seen that the edges e_1 and e_2 do not belong to any perfect matching. Therefore G'' has a perfect matching if and only if G' has a perfect matching.

On the other hand, it is possible to construct a perfect matching for G'' by enlarging any matching M' on G' by edges $\{s_j, x_1\}, \{t_j, y_2\}$ and natural enlargements on $H_j(x_1, x_2) - \{x_1\}$ and $H_j(x_1, x_2) - \{x_2\}$. This construction can also be done in AC^0.

From a perfect matching in G'' a perfect matching in G' can be constructed in AC^0. $\square$

The immediate class above the 3-regular 2-connected graphs is the class of 3-regular graphs (which are not necessarily 2-connected). To complete the proof of the main theorem we have to show the following:

Lemma 3. The existence and the construction problem for the perfect matching for 3-regular graphs is AC^0-equivalent with the existence and the construction problem for a general matching problem respectively.

PROOF. Consider any graph G of maximal degree 3 which is 2-connected. For each vertex u of G of degree 2 let u_1 and u_2 be the corresponding vertices in G_1 and G_2, respectively.
Join u_1 and u_2 by the following graph H (see Figure 4).

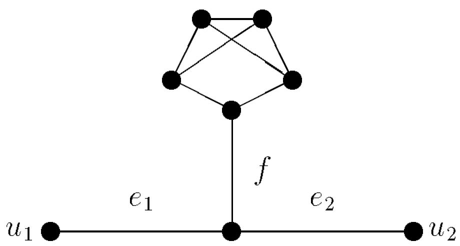

Figure 4:

Call the resulting graph G'. Then the edge f belongs to any perfect matching and therefore e_1 and e_2 both belong to no perfect matching of G'. G' is 3-regular and G' has a perfect matching if and only if G has a perfect matching.

It is easily seen that we can construct a perfect matching in G from a perfect matching in G' in AC^0.

$\square$

5 Final Remarks.

The construction of a perfect matching for bridgeless 3-regular graphs in parallel remains an open problem. We conjecture the problem lies in NC. We also refer to [KUW 2]. This paper deals with the question of equivalence of existence and construction problems.

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