There Is No Polynomial Deterministic Space Simulation of Probabilistic Space with a Two-Way Random-Tape Generator *

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Abstract

We prove there is no polynomial deterministic space simulation for two-way random-tape probabilistic space ($\mathrm{Pr}_2\mathrm{SPACE}$) (as defined in [BCP 83]) for all functions $f : \mathbb{N} \to \mathbb{N}$ and all $\alpha \in \mathbb{N}$, $\mathrm{Pr}_2\mathrm{SPACE}(f(n)) \not\subseteq \mathrm{DSPACE}(f(n)^\alpha)$. This is answer to the problem formulated in op cit., whether the deterministic squared-space simulation (for recognizers and transducers) generalizes to the two-way random-tape machine model. We prove, in fact, a stronger result saying that even space-bounded Las Vegas two-way random-tape algorithms (yielding always the correct answer and terminating with probability 1) are exponentially more efficient than the deterministic ones.

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1 Introduction

Jung (1981) and Borodin, Cook, and Pippenger (1983) prove that both the probabilistic acceptors and transducers working in space $f(n) \geq \log n$ can be simulated in deterministic $f(n)^2$ space. The definition of probabilistic Turing machines uses a one-way read-only random tape. The model of probabilistic machine [Gi 77] may be reviewed as a deterministic machine with a one-way only access to the random bits sequence. A two-way random tape proposed in [BCP 83] allows multiple access to the random bits sequence which is stored on the two-way read-only tape. The problem posed in [BCP 83] whether the $f(n)^2$ deterministic space simulation holds also for the two-way random-tape ($\mathrm{Pr}_2\mathrm{SPACE}(f(n))$).

Let $\Psi \subseteq \Sigma^* \times \{0,1\}^\omega$ be a binary predicate, where $\Psi(x,y)$ is computed by a deterministic machine M with two two-way read-only input tapes. If M stops on an initial segment of Y, then $\Psi(x,y)$ is defined. $x \in \Sigma^*$ is recognized by M if and only if $\Pr\{\Psi(x,y) = \text{true}\} > \frac{1}{2}$. We call M a probabilistic machine (over the alphabet Σ) with two-way random tape. Let $L_M \subseteq \Sigma^*$ denote the set recognized by M. If M is $S(|x|)$ space bounded, then L_M belongs to the two-way random-tape probabilistic space $S(n)$, $L_M \in \mathrm{Pr}_2\mathrm{SPACE}(S(n))$. If in addition M is $T(|x|)$ time bounded, then $L_M \in \mathrm{Pr}_2\mathrm{TISP}(T(n), S(n))$. We say that L_M belongs to the two-way Las Vegas [BGM 82] space $S(n)$, $L_M \in \Delta_2\mathrm{SPACE}(S(n))$, if for all $x \in \Sigma^*$ either $\Pr\{\Psi_M(x,y) = \text{true}\} = 1$ or $\Pr\{\Psi_M(x,y) = \text{false}\} = 1$.

We prove that the class of $\log F(n)$ space bounded Las Vegas algorithms with two-way random-tape (terminating with probability 1 and yielding always the correct result) denoted by $\Delta_2\mathrm{SPACE}(\log f(n))$ (time bounded Las Vegas algorithms are defined in [AM 77]; [BGM 82] are as powerful as $\mathrm{DSPACE}(f(n))$. Therefore there is no polynomial simulation for this class, which answers the problem of [BCP 83].

2 Remarks

1. This result is related to the recent result of Savitch and Dymond ([SD 84]) that "consistent" NSPACE is exponentially more powerful than DSPACE. The similarity becomes clear, if the reset mechanism in the original definition of consistent NSPACE is replaced by a two-way tape, of which the initial nondeterministic choices are stored. The proof of our Theorem 2 can be applied to this case.

2. The model of a probabilistic machine with two-way random tape may be viewed

as a deterministic machine with a random oracle stored on a two-way tape. The oracle tape records the outcome of an infinite sequence of independent unbiased coin tosses. The classical model of Gill ([Gi 77]) may be viewed as a deterministic machine with a random oracle stored on a one-way tape. The classical oracle machine ([BG 81]) is a deterministic machine with oracle stored on a derive resembling random-access store rather than tape (i.e., the question must be written on a query tape within the space bound). Denote by $\mathrm{DSPACE}^{(A)_2}(f(n))$ the class of sets recognized by $f(n)$ space bounded deterministic Turing machines with oracle A stored on a two-way tape. Then, with probability 1 (i.e., for almost all oracles), $\mathrm{DSPACE}^{(A)_2}(f(n)) \not\supseteq \Delta_2\mathrm{SPACE}(f(n))$ (the inequivalence results from the fact that, with probability 1, $A \notin \Delta_2\mathrm{SPACE}(f(n))$).

3 Results

Theorem 1. *For every function $f : I\!N \to I\!N$,*

$$\bigcup_{k \in I\!N} \Delta_2\, TISP(2^{2^{2^{k\cdot\log f(n)}}}, \log f(n)) \supseteq DSPACE(f(n))\,.$$

Corollary. *For every function f,*

$$Pr_2SPACE(\log f(n)) \supseteq \Delta_2 SPACE(\log f(n)) \supseteq DSPACE(f(n))\,.$$

Corollary (Problem of [BCP 83]).

$$\mathrm{Pr}_2\mathrm{SPACE}(f(n)) \not\subseteq \mathrm{DSPACE}(f(n)^2)\,.$$

Proof of Theorem 1. Suppose $\mathcal{T}$ is a $f(n)$ space bounded deterministic Turing machine with one work tape. Suppose that $\mathcal{T}$ stops on every input (see [Si 80]).

For $x \in \Sigma^*$, $\mathrm{comp}_{\mathcal{T}}(x) \in \bar{\Sigma}^*$ will denote the computation of $\mathcal{T}$ over x (not recording the input or input position). The probability that the random tape will contain as a subsequence $\cent\ \mathrm{comp}_{\mathcal{T}}(x)\,\$$, $x \in \Sigma^*$ (encoded as a binary sequence), is equal to 1. On the other hand, the set $\{(x, u\ \cent\ \mathrm{comp}_{\mathcal{T}}\ \$\ v) \mid x \in \Sigma^*,\ u, v \in \bar{\Sigma}^*\}$ is recognized by a $\log f(n)$ bounded deterministic Turing machine $\mathcal{M}$ with two input tapes (only the position in the current storage-configuration of $\mathcal{T}$ must be stored).

Take now this machine $\mathcal{M}$, put it on the random tape and let it search for ¢ $\mathrm{comp}_{\mathcal{T}}(x)$ \$. This string will appear on the random tape with probability 1. Thus $\mathcal{M}$ stops with probability 1 and gives the correct result (according to the halting configuration in $\mathrm{comp}_{\mathcal{T}}(x)$). The expected time for the simulation lies in

$$\bigcup_k (2^{k\cdot|comp_{\mathcal{T}}(x)|}) \leq \bigcup_k (2^{f(|x|)\cdot 2^{k\cdot f(|x|)}}) \leq \bigcup_k (2^{2^{2^{k\cdot \log f(|x|)}}}) .$$

Theorem 1 is valid also for transducers; in this case $\mathcal{M}$ begins outputing after it has found and verified $\mathrm{comp}_{\mathcal{T}}(x)$. □

Theorem 2. *For every function f,*

$$\Delta_2 SPACE(f(n)) \subseteq \bigcup_k SPACE(n^4 \cdot 2^{k\cdot f(n)}) .$$

Corollary. *If $f(n) \geq \log n$, then*

$$\Delta_2 SPACE(f(n)) = \bigcup_k DSPACE(2^{k\cdot f(n)}) .$$

In particular,

$$\Delta_2 SPACE(\log n) = PSPACE.$$

Proof of Theorem 2. Let $\mathcal{M}$ be an $f(n)$ bounded Δ_2 machine. A configuration of $\mathcal{M}$ contains the position on the input and the content of the work tape (but not the position on the random tape). The number of configurations accessible on input x is bounded by $|x| \cdot 2^{k\cdot f|x|}$.

$\mathcal{M}$ is simulated by a Δ_1-machine $\mathcal{T}$ (i.e. with one-way random- tape) in the same way as a two-way finite automaton is simulated by a one-way FA (see [HU 79]). It holds a table which says for each pair of configurations: if $\mathcal{M}$ is in configuration c and goes left (on the random tape) then it can (or cannot) come back in configuration c'. In addition it is stored whether or not $\mathcal{M}$ starting in configuration c can go left and never come back (in this case it is stored whether $\mathcal{M}$ accepts or rejects).

It is easy to see that $\mathcal{T}$ uses $(|x| \cdot 2^{k\cdot f|x|})^2$ space for two such tables and that these tables are sufficient to determine whether $\mathcal{M}$ stops, and if it stops, to determine the decision. Since $\mathcal{M}$ never gives a wrong result, $\mathcal{T}$ accepts the same sets as $\mathcal{M}$. Since $\Delta_1\mathrm{SPACE}(f(n)) \subseteq \mathrm{PrSPACE}(f(n)) \subseteq \mathrm{DSPACE}(f(n)^2)$ [BCP 83] $\mathcal{T}$ can be simulated by a deterministic machine in $O(|x|^4 \cdot 2^{4k\cdot f|x|})$ space. □

We were not able to extend the upper bound of Theorem 2 to the case of probabilistic machines with non-zero error probability. It is even not known whether or not $\mathrm{Pr}_2\mathrm{SPACE}$ is Blum complexity measure [Bl 67].

4 Open Problem

Is there a recursive function h, such that for every f

$$\mathrm{Pr}_2\mathrm{SPACE}(f(n)) \subseteq \mathrm{DSPACE}(hf(n))\ ?$$

Is every set recognized by a probabilistic *finite* automaton with two-way random-tape recursive, i.e., $\mathrm{Pr}_2\mathrm{SPACE}(O(1)) \subseteq \mathrm{DSPACE}(h(n))$ for some recursive h?

(By [KV 84] the set of *computations* can be recognized by probabilistic finite two-way automata with one-way random-type and bounded error probability).[1]

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[1] *Note in proof.* Meanwhile the authors were able to solve this problem. The first function h mentioned above is in fact recursive and $2^{O(n)}$ and the second is $n^2 \log^2 n$. Therefore $\mathrm{Pr}_2\mathrm{SPACE}$ is a Blum complexity measure.

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