Probabilistic NC^1-Circuits Equal Probabilistic Polynomial Time

Marek Karpinski *†
Rutger Verbeek *

Abstract We prove that probabilistic NC^1 ($PrNC^1$) circuits (i.e. uniform log-depth poly-size circuits with unbounded error probability) are computationally exactly as powerful as probabilistic polynomial time. This entails that the probabilistic NC^k-hierarchy collapses at the NC^1 level; if unbounded fan-in is allowed it collapses even at the level 0. As a side effect we prove the identity $PrNC = Pr_2SC = Pr_2SC^1$ (Pr_2SC^k meaning simultaneous polynomial time and $\log^k n$ space bounded machines with two-way random tape [KV 84]). The central problems in computational complexity theory are whether $NC = P$ [Co 83], $NC^2 = NC$ and $SC = NC$ [Co 79, Ru 81] and the most classical problem whether LOGSPACE = P. Surprisingly the results of the present paper and [KV 84] give affirmative answer to all these questions in the probabilistic case.

---

* Dept. of Computer Science, University of Bonn, Wegelerstraße 6, 5300 Bonn 1 W.-Germany (mailing address)

† Dept. of Computer Science, University of Dortmund, 4600 Dortmund 50, W.-Germany

1. Probabilistic uniform circuits

The reader is referred to [Co 83] for an extended exposition on uniform circuits. The main definitions are given below.

A circuit C with n inputs is a finite directed acyclic graph, such that each node has a label from $\{x_1,\ldots,x_n\} \cup \{\wedge,\vee,\neg\}$. A node labelled x_i has indegree (fan-in) 0 and is called an input node. A node v with label from $\{\wedge,\vee\}$ must have indegree 2, whereas v with label $\neg$ has indegree 1. Exactly one node does have outdegree (fan-out) 0; we call it the output node y. The fan-out of the other nodes is unbounded.
The size of C ($s(C)$) is the number of nodes in C, the depth $d(C)$ is the length of the longest path in C. Every 0-1 assignment to the input nodes (interpreted as boolean variables) yields unique 0-1 assignment to all the remaining nodes (including y). In this way one defines a boolean function $f_C : \{0,1\}^n \to \{0,1\}$, called the function computed by C.

A function $f : \{0,1\}^* \to \{0,1\}$ is computed by a circuit family $\langle C_n \rangle$, $n \in \mathbf{N}$, if for every n, $f_{C_n} = f \restriction \{0,1\}^n$. A circuit family $\langle C_n \rangle$ is called uniform, if C_n can be constructed from n in $O(\log n)$ space [Bo 77, Ru 81].
NC^k is the class of all functions computable by a uniform circuit family with $s(C_n) = n^{O(1)}$ and $d(C_n) = O(\log^k n)$, $NC = \bigcup_k NC^k$.
We shall extend the notion of a circuit to circuits with unbounded fan-in for 'AND' and 'OR' gates [SSF 81]. The corresponding classes of functions will be denoted by QNC^k and QNC.

A probabilistic circuit [Co 83] is a circuit C with ordinary inputs $x_1,\ldots,x_n$ and designated coin-tossing inputs $z_1,\ldots,z_m$. The probability that the output y is one (on input $x_1,\ldots,x_n$) is the fraction of input bit-vectors $z_1,\ldots,z_m$ for which $f_C(x_1,\ldots,x_n, z_1,\ldots,z_m) = 1$. We say a function f is probabilistically computed by $\langle C_n \rangle$, if for all n and all $x_1,\ldots,x_n$
$\Pr\{f_C(x_1,\ldots,x_n, z_1,\ldots,z_m) = f(x_1,\ldots,x_n)\} > \frac{1}{2}$. (When $\frac{1}{2}$ in the definition above is replaced by $\frac{3}{4}$, f is Monte-Carlo computable by C [Co 83]).

$PrNC^k$ is the class of all functions probabilistically computable by an uniform circuit family with depth $O(\log^k n)$ and polynomial size, $PrNC = \bigcup_k PrNC^k$; for unbounded fan-in $PrQNC^k$ and $PrQNC$ is defined analogously. (The class $PrNC$ is the probabilistic version of S. Cook's Monte-Carlo RNC-class [Co 83].)

2. Uniform circuits and two-way random generators

For an exact definition of two-way random-tape and the corresponding complexity classes see [KV 84].
Informally, a language in $Pr_2SPACE(f(n))$ is recognized by a probabilistic $f(n)$-space bounded machine with two-way access to a random sequence. The following depends on the fact that circuits have multiple access to the random input.

Theorem 1 [KV 84] Probabilistic machines with two-way random-tape that are simultaneously $\log n$ - space and polynomial-time bounded are as powerful as those without restriction on space:
$Pr_2SC^1 = PP$.

The proof of Theorem 1 is based on the following construction (Lemmas 1 and 2) which is adapted from the proof of Lemma 5 of [KV 84] and modified now for application in uniform circuits.

Let M be a probabilistic strictly n^k-time bounded one-tape machine. (For every $f \in PP$ there exists k, such that f is strictly n^k-time computable [Gi 77]). Denote by $comp_M(x)$ the set of M-computations on input x encoded by $\not\$ c_o \alpha_o \$ c_1 \alpha_1 \$ \ldots \$ c_{n^k} \not\$ \in \Sigma^*$, where the c_i's are encodings of IDs padded with blanks to exactly the same length $n+n^k$ and the α_i's are the random bits of the computation, such that $c_i \vdash_M c_{i+1}$ for the random bit α_i. A stopping ID $c_i, i < n^k$, is identically repeated up to the step n^k with arbitrary random bits α_i.
We encode now the computations in binary using a coding function $h : \Sigma \to \{0,1\}^l$ for an appropriate l. Denote $bincomp_M(x) = h(comp_M(x))$ (for h naturally extended over Σ^*).

Lemma 1 Given an arbitrary probabilistic strictly n^k-time bounded one-tape machine M, there exists a deterministic log-space bounded machine $\overline{M}$, such that $\overline{M}$ computes the function $f : \Sigma^* \times \{0,1\}\{0,1\}^* \to \{0,1\}$:

$$f(x,\alpha y) = \begin{cases} 1 & \text{if } y \in bincomp_M(x) \text{ and } h^{-1}(y) \text{ is accepting} \\ 0 & \text{if } y \in bincomp_M(x) \text{ and } h^{-1}(y) \text{ is rejecting} \\ \alpha & \text{if } y \notin bincomp_M(x) \end{cases}$$

for $x \in \Sigma^*$, $\alpha \in \{0,1\}$, $y \in \{0,1\}^*$ ($h : \Sigma^* \to \{0,1\}^*$ as above).

Proof Standard construction as for deterministic machines (cf. [HU 79]).

□

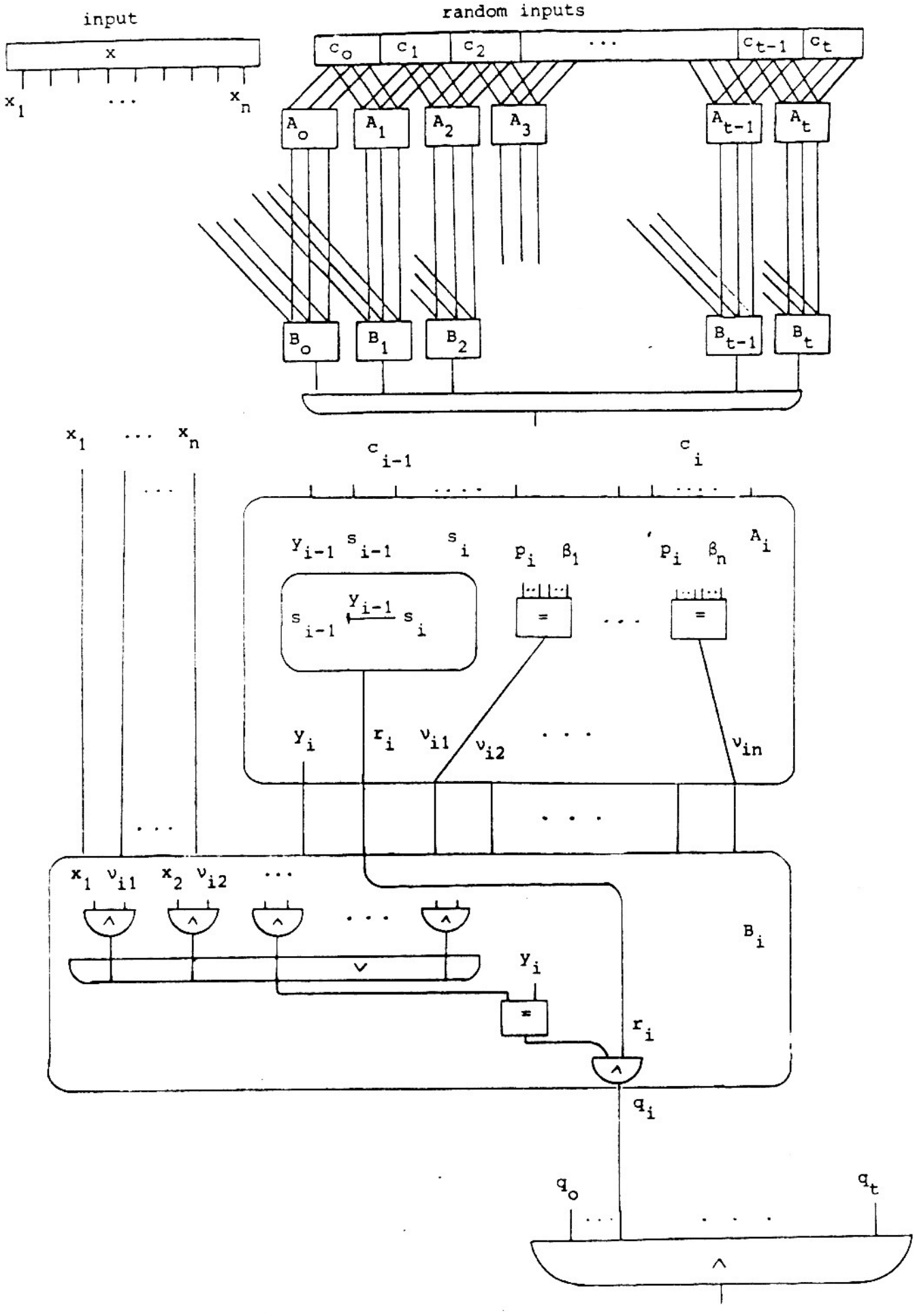
For a deterministic $\log n$-space bounded machine M with binary input $\text{bincomp}_M(x)$ will denote now the binary encoding of the computation of M on x, such that the codes of single configurations have the same length $l \cdot \log n$ for an appropriate fixed l. We denote by c_i the code of the ith configuration consisting of s_i, p_i, y_i denoting s_i, the contents of the worktapes and the state, p_i, the binary code of the input position, y_i, the input symbol (i.e. $y_i = x_{p_i}$). Define $\text{TEST}_M(x,y) = 1$ if $y = \text{bincomp}_M(x)$ and 0 if $y \neq \text{bincomp}_M(x)$.

Lemma 2 Given arbitrary $\log n$-space bounded machine M with binary input, $\text{TEST}_M(x,y)$ is computed by uniform poly-size constant-depth unbounded fan-in circuits,
$\text{TEST}_M \in \text{QNC}^0$.

Proof (cf. Figure 1)

Let t denote an upper bound on the running time of M, i.e. $t = n^k$ for an appropriate k depending on M.

The circuits $A_i (1 \leq i \leq t)$ compare the configurations c_{i-1} and c_i and generate a "correctness bit" r_i, which is set to 1 iff $(s_{i-1}, p_{i-1}) \xrightarrow{y_{i-1}} (s_i, p_i)$. A_o checks, whether c_o is a legal initial configuration. Furthermore A_i $(o \leq i \leq t)$ outputs y_i and an unary representation of p_i, i.e. $\nu_{ij} = 1$ iff $j = p_i$. To do this, it compares (in parallel) p_i with β_j $(1 \leq j \leq n)$, β_j denoting the binary code of j.
The circuits B_i $(o \leq i \leq t)$ select the p_i's input bit (using ν_{ij}'s), compare it with y_i and set the output bit q_i to 1 iff $r_i = 1$ and $y_i = x_{p_i}$. If all q_i's are 1, the whole circuit outputs 1.

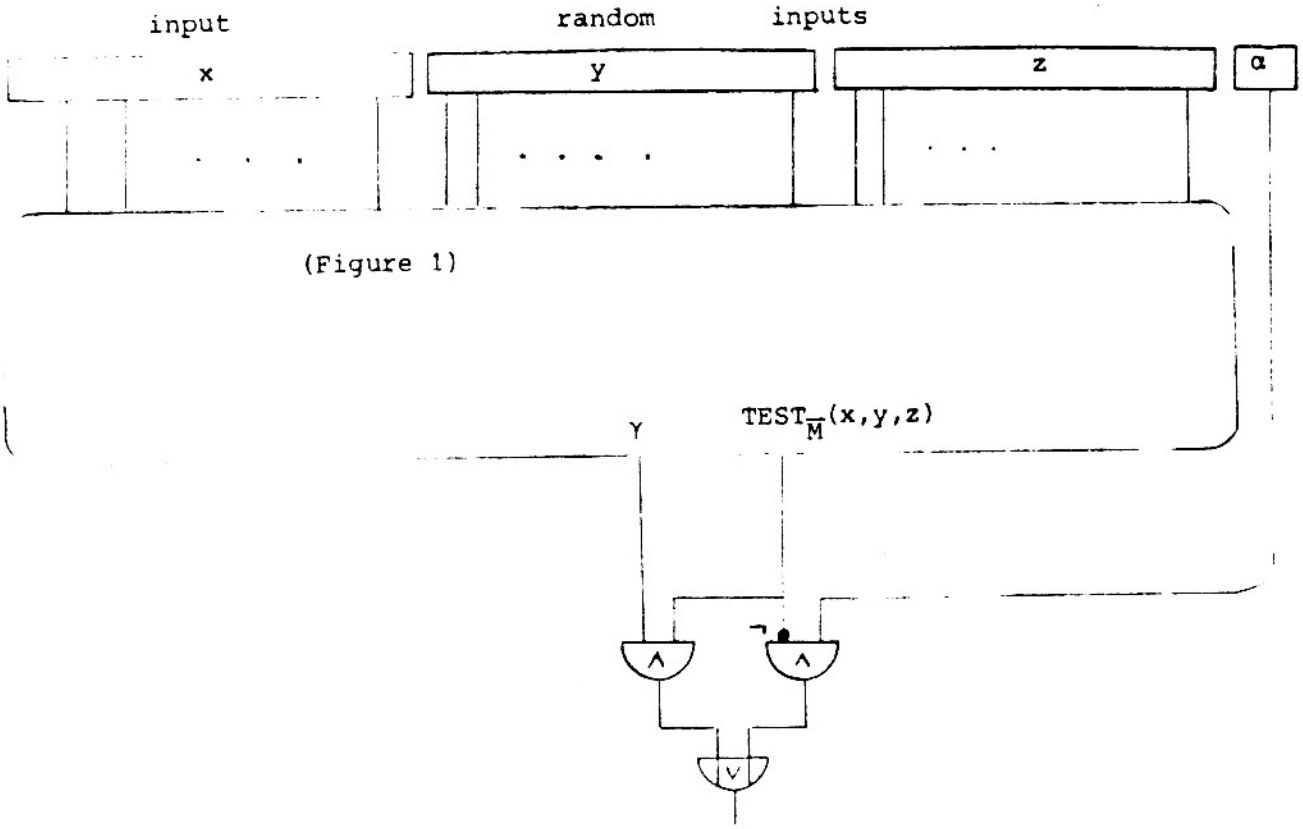

input
x
x_1 ... x_n
random inputs
c_o c_1 c_2 ... c_{t-1} c_t
A_o A_1 A_2 A_3 A_{t-1} A_t
B_o B_1 B_2 B_{t-1} B_t
x_1 ... x_n
c_{i-1} c_i
y_{i-1} s_{i-1} s_i p_i β_1 p_i β_n A_i
s_{i-1} y_{i-1} s_i
=
y_i r_i ν_{i1} ν_{i2} ν_{in}
x_1 ν_{i1} x_2 ν_{i2} ...
B_i
y_i
r_i
q_i
q_o q_t

Since the circuits A_i have only $O(\log n)$ inputs and $O(n)$ outputs, the standard depth 3 CNF-representation of their functions have polynomial size; since the B_i's have constant depth and polynomial size, this is true for the whole circuit.
Uniformity of our circuit family is guaranteed by the fact that A_i's $(1 \leq i \leq t)$ are all identical and the same holds for all B_i's.

Theorem 2 Any boolean function computed by a polynomial-time bounded machine is computed by some uniform family of probabilistic circuits $\langle C_n \rangle$ with polynomial size, constant depth, and unbounded fan-in:

$\mathrm{PrQNC}^0 \supseteq \mathrm{PP}$.

Proof (cf. Figure 2)
Let M be a probabilistic poly-time machine, $\overline{M}$ the log-space machine of Lemma 1. The circuit C_n has inputs x,y,z,α (x is an ordinary input of size n; y,z are random inputs of appropriate polynomial size; α is a single random bit).



(Figure 1)

Figure 2

Using the circuit of Lemma 2 it computes $TEST_{\overline{M}}(xy,z)$ and in addition the output of $\overline{M}$ called γ (which can be found at a fixed position in z, if $z = bincomp_{\overline{M}}(xy)$. If $TEST_{\overline{M}}(xy,z) = 1$, then C_n outputs γ, otherwise C_n outputs α.

Let p denote the probability that M outputs 1 on x, $n := |x|$.
$q := Pr\{y \in bincomp_M(x) \text{ and } z = bincomp_{\overline{M}}(x,y)\}$.
Then $Pr\{C_n \text{ outputs } 1\} = p \cdot q + \frac{1}{2} \cdot Pr\{y \notin bincomp_M(x) \text{ or } z \neq bincomp_{\overline{M}}(xy)\}$

$$= p \cdot q + \frac{1}{2}(1-q) = \frac{1}{2} + (p - \frac{1}{2}) \cdot q > \frac{1}{2} \iff p > \frac{1}{2}$$

$\iff$ M accepts x.

□

PC will stand for the class of boolean functions computed by uniform polynomial-size circuits.

Lemma 3 The probabilistic uniform poly-size circuit class is included in probabilistic polynomial time,
PrPC $\subseteq$ PP.

Proof Given uniform family of probabilistic circuits $<C_n>$, the simulating poly-time bounded machine constructs the circuit C_n in its memory, using its random generator to assign values to the random inputs of C_n. Since the circuit with the random bits fixed behaves deterministically we can simulate it in deterministic polynomial time (cf. [Bo 77]). Since the random pads required for the circuit and the machine have the same length, the probabilities for accepting and rejecting are identical in both models.

□

Theorem 3 The following classes of 0-1-valued functions are all equivalent:

(1) $PrNC^1$ (probabilistic log depth)
(2) PrNC (probabilistic poly-log depth, poly-size)
(3) $PrQNC^0$ (probabilistic constant depth, poly-size)
(4) Pr_2SC^1 (probabilistic log-space poly-time with two-way random tape, cf. [KV 84])
(5) PrPC (probabilistic poly-size)
(6) PP (probabilistic poly-time)

$PrNC^1 = PrNC = PrQNC^0 = Pr_2SC^1 = PrPC = PP.$

Proof The equalities follow from Theorem 1, Theorem 2, Lemma 3 and the fact that for all k, $PrQNC^k \subseteq PrNC^{k+1}$ (decompose a gate with unbounded fan-in $n > 2$ into a $\log n$-depth circuit, cf. [Co 83]).

We define the classes of probabilistic k-bounded alternation-depth circuits as uniform circuit families with $O(\log^k n)$ levels of AND and OR gates with unbounded fan-ins and negations pushed to the inputs (cf. [Co 83]). Denote the corresponding classes of functions by $PrADC^k$, $k = 1, \ldots$, $PrADC = \bigcup_k PrADC^k$. □

Theorem 4 The probabilistic alternation-depth hierarchy collapses at level 1, $PrADC^1 = PrADC = PP$.

Proof By Theorem 2 $PP \subseteq PrQNC^0$ and this is contained in $PrADC^1$.
On the other hand $PrADC^k \subseteq PrQNC^k$. □

It is well known [BG 81], [AB-O 84] that nonuniform deterministic poly-size circuits are as powerful as Monte-Carlo ones. By [AB-O 84] the same is true for corresponding deterministic and Monte-Carlo classes of unbounded fan-in. By [FSS 84] and Theorem 3, the class of uniform probabilistic circuits of constant depth ($PrQNC^0$) is not included in the class of nonuniform deterministic polynomial size circuits of constant depth (the parity function is in P and therefore in $PrQNC^0$, but not in nonuniform QNC^0).

Theorem 5 $PrQNC^0 \not\subseteq$ nonuniform QNC^0. □

3. Conclusion

There are natural functions in $PrQNC^0$, which are not in $RQNC^0$, e.g. majority and parity. The positive answer to the question "are the probabilistic uniform log-depth circuits equivalent to the Monte-Carlo uniform log-depth circuits" would require a breakthrough in complexity theory since $PrNC^1 \neq RNC^1 \subseteq BPP$ unless Monte-Carlo poly-time equals probabilistic poly-time. One level higher a negative answer to the same question (with $\log n$ replaced by $\log^2 n$), i.e. $PrNC^2 \neq RNC^2$ would imply probabilistic LOGSPACE is unequal to probabilistic polynomial time.

Finally we indicate another application of our result towards probabilistic versions of the parallel WRAMs of [CSV 82]: any such (both deterministic and probabilistic) WRAM with a polynomial number of processors can be simulated by some PrWRAM with a polynomial number of processors in $\log n$ parallel time.

References

[AB-O 84] Ajtai, M., and Ben-Or, M.,
A theorem on probabilistic constant depth computations,
Proc. 16th ACM STOC (1984), pp. 471-474

[Bo 77] Borodin, A.,
On relating time and space to size and depth,
SIAM Journal on Computing 6 (1977), pp. 733-744

[BCP 83] Borodin, A., Cook, S., and Pippenger, N.,
Parallel computation for well endowed rings and space-bounded probabilistic machines,
Information and Control 58 (1983), 96-114

[BG 81] Benett, C., and Gill, J.,
Relative to a random oracle A, $P^A \neq NP^A \neq co\text{-}NP^A$ with probability 1,
SIAM Journal on Computing 1o (1981), pp. 96-114

[Co 83] Cook, S.
The classification of Problems which have fast parallel algorithms,
Proc. Foundations of Computation Theory, Springer LNCS 158,
(1983), pp. 78-93

[CSV 82] Chandra, A.K., Stockmeyer, L.J., and Vishkin, U.,
A complexity theory for unbounded fan-in parallelism,
Proc. 23rd IEEE FOCS (1982), pp. 1-13

[FSS 81] Furst, M., Saxe, J.B., and Sipser, M.,
Parity, Circuits, and the polynomial time hierarchy,
Proc. 22nd IEEE FOCS (1981), pp. 26o-27o

[Gi 77] Gill, J.,
Computational complexity of probabilistic Turing machines,
SIAM Journal on Computing 6 (1977), pp. 675-694

[HU 79] Hopcroft, J., and Ullman, J.,
Introduction to automata theory, languages, and computation,
Addison-Wesley, Reading (1979).

[KV 84 a] Karpinski, M., and Verbeek, R.,
There is no polynomial deterministic space simulation of probabilistic space with two-way random-tape generator,
Interner Bericht I 4 des Instituts für Informatik, Universität Bonn (1984), submitted to Information and Control

[KV 84 b] Karpinski, M., and Verbeek, R.,
On the power of two-way random generators and the impossibility of deterministic poly-space simulation,
Interner Bericht I/5 des Instituts für Informatik, Universität Bonn (1984)

[Pi 79] Pippenger, N.,
On simultaneous resource bounds (preliminary version),
Proc. 2oth IEEE FOCS (1979), 3o7-311

[Ru 81] Ruzzo, W.L.,
On uniform circuit complexity,
Journal of Computer and System Sciences 22 (1981), pp. 365-383

[Si 83] Sipser, M.,
A complexity theoretic approach to randomness,
Proc. 15th ACM STOC (1983), pp. 33o-335

[St 83] Stockmeyer, L.J.,
The complexity of approximate counting,
Proc. 15th ACM STOC (1983), pp. 118-126